The Divisibility of Normal Chern Numbers

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Abstract

Following the work of Rees, Thomas and Barton on the divisibility properties of certain normal chern numbers some chern numbers of the Milnor-Novikov generators of the cobordism ring are examined. The divisibility properties, at least up to 2-torsion, of these chern numbers are computed and these properties are then used to construct the manifolds whose chern numbers realize the minimum divisibility. As an example of how these direct methods can be employed an observation of Libgober and Wood is verified and improved upon.

Odd torsion is also examined. It is observed that a proof from the work of Barton and Rees is incomplete and that proof is duly completed. Symmetric functions are introduced to form natural coefficients for a formal sum of the chern numbers of a manifold. Using this construction a bound on the primes contained in the hcf of any chern number is obtained, where this bound is dependant upon the length of the partition which indexes the chern number. A systematic method for lowering this bound (often eliminating odd torsion completely) for particular examples is demonstrated.

As a digression the link between chern numbers and symmetric functions is examined in its own right. In particular the combinatorial side is addressed through the generalization of a partition of a number to a partition of a set. The general case of an arbitrary chern number of an arbitrary cross product of projective spaces is considered in detail and a general formula is obtained using the language of the lattice of partitions of a set. Examples to demonstrate the viability of this approach are presented.

Acknowledgements

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- and most of all to Moyra, who was just always there.

Contents

Chapter 1

1.1 Introduction

In [15], Rees and Thomas discuss the question of whether a complex manifold with an isolated singularity can be made smooth. Previous work of Thom has suggested that this is, in fact, a cobordism problem, with obstructions dependent upon the cobordism type of the intersection of the manifold with the surface of a ball forming a sufficiently small neighbourhood of the singularity in the ambient space.

In [15] and [14] these obstructions, which lie in homotopy groups of $MU(n)$, the Thom space of the universal bundle over the classifying space $BU(n)$, are shown to be closely related to the divisibility of certain normal chern numbers. The relevant homotopy groups themselves have summands that are cyclic of order 2^ρ where ρ is determined by the divisibility of all chern numbers. It is unrealistic to expect that all these obstructions could be calculated but it can be seen that knowledge of their divisibility, and especially a calculation of the highest common factor of these chern numbers over all manifolds, is very useful in this context. Similar problems occur in other works in this field, notably in [6], [7] and in a more general setting in [5] and other works of Holme, who invites a more detailed examination of normal chern numbers.

Some steps have been taken to examine the divisibility of the chern numbers that appear in this way. A treatment of some low dimensional cases is made by Liulevicius in [7], while Rees, Thomas and Barton cover the cases that occur explicitly in the above in [12] and [2].

In this thesis a comprehensive study is made of the divisibility properties of normal chern numbers. The work of Rees, Thomas and Barton is verified in a constructive manner and manifolds are constructed which realize these divisibility properties.

In the general case a bound is obtained on the primes which will occur in the h.c.f., and methods for dealing with particular cases are dealt with by example.

An additional chapter is included where the chern numbers of cross-products of projective spaces are studied in detail and a general formula is obtained in a purely algebraic fashion, using a generalization of symmetric functions presented by Doubilet in [3].

The first chapter of the text is chiefly functional, setting up the notation which is to be used in Chapters 2 and 3. It also, in Section 4, presents the initial results, some of which are well-known, all of which are straightforward, that will be used to effect in the remainder of the thesis.

Chapter 2 is calculational. It begins by examining the Milnor-Novikov generators of the cobordism ring, $\{K^1, K^2, K^3, \ldots\}$, which are given an exact definition by Milnor and Stasheff in [10].

The divisibility properties, at least up to 2-torsion, of some chern numbers of these generators are then methodically computed in Sections 2, 3, 4 and 5, and these properties are then used to constuct manifolds which realize the minimum divisibility in Section 6. The methods of this latter section demonstate clearly that the key to understanding more complicated chern numbers is in realizing how they are built up, using the cross product, from simpler chern numbers and as such provides motivation, with hindsight, for spending so much time and effort in calculating the simple cases.

The final section of this chapter serves mainly as an example of how the direct methods used in the previous sections can be employed for the evaluation of particular examples. Here an observation of Libgober and Wood from [6] is verified and improved upon.

The third chapter turns to the question of odd torsion. Some work in this direction has been undertaken in [2] where, by examining two examples of matrices of normal chern numbers, odd torsion is effectively eliminated from the study of the divisibility of a particular class of chern numbers. In the first section of this chapter it is observed that the proof of the second example in [2] is incomplete and that proof is duly completed. The remainder of the chapter can be viewed as a generalization of the ideas inherent behind the first example of this paper.

It is at this stage that symmetric functions require to be introduced to facilitate the details of this generalization. It is observed that the monomial symmetric functions form natural coefficients for a formal sum of the chern numbers of a manifold and so they provide the ideal basis from which to work. Using this construction we obtain a bound on the primes contained in the h.c.f. for any chern number, where this bound is dependant upon the length of the partition which indexes the chern number.

We also present, by example, a method for lowering this bound (often eliminating odd torsion completely) for particular cases.

Chapter 4 is to be considered as a digression from what has gone before. In this chapter the link between chern numbers and symmetric functions, established as a working tool in Chapter 3, is examined in its own right. In particular, the combinatorial side is addressed from the point of view of Doubilet, as expressed in [3], which is to generalize the concept of a partition of a number to that of a partition of a set.

The general case of an arbitrary chern number of an arbitrary cross product of projective spaces is considered in detail and we obtain a general formula in terms of the language of the lattice of partitions of a set. The results obtained have a naturality and elegance not present in more direct methods. It is shown that it is possible to make direct calculations using this approach and it is demonstrated by example that such calculations, although lengthy, are systematic and so lend themselves readily to the computation of high-dimensional examples.

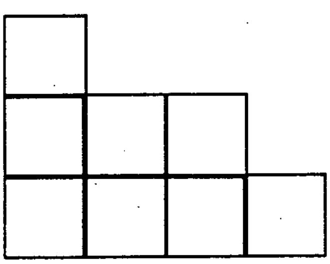
1.2 Partitions

In this section we shall introduce partitions and ordered partitions. Notation varies considerably in the literature, we shall stick mainly to the notation used in [1] and [10] with slight differences, especially to the latter reference, mainly to accomodate further properties we shall use in Chapter 4. In particular proofs and more details of the following elementary facts can be found in these references, as well as in many others.

A partition $I = (i_1, \ldots, i_r)$ is a sum $i_1 + \cdots + i_r = m$, where each number $i_1, \ldots, i_r$ is a positive integer. At times we will abuse this notation slightly and refer to I as the partition $(i_1 + \cdots + i_r = m)$. We shall call two partitions I and J equal if one is a permutation of the other, and so we will normally keep to the convention that $i_1 \geq \cdots \geq i_r > 0$. Note we can distinguish between the number m and the single-term partition (m).

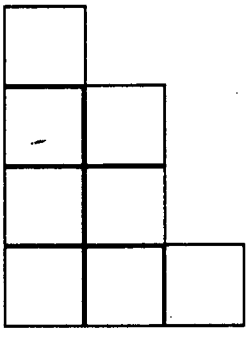
Given a partition $i_1 + \cdots i_r = m$ we will refer to the obvious picture:

Ex. $I = (3, 2, 2, 1)$



Hence we will define the length of I, $l(I) = r$; the height of I, $h(I) = i_1$ and the weight of I, writing $I \vdash m$.

If we reflect this picture through the line $y = x$ we get the picture corresponding to the *dual partition* $I' = (i'_1 + \cdots + i'_t = m)$. For example, the dual of $(3, 2, 2, 1)$ is $(4, 3, 1)$.



In fact we have that $i'_k := \#\{i_j \geq k\}$. It is clear that $l(I) = h(I')$ and vice-versa.

Given two partitions $I = (i_1 + \cdots + i_r = m)$ and $J = (j_1 + \cdots + j_s = n)$ we can *juxtapose* them together to get $I \wedge J = (i_1 + \cdots + i_r + j_1 + \cdots + j_s = m + n)$, where $l(I \wedge J) = r + s$ and $h(I \wedge J) = max(i_1, j_1)$.

At times it will be useful to have a different notation for a partition $I \vdash m$. We will also describe I to be the partition $1^{f_1}.2^{f_2}.\ldots.m^{f_m}$, where '1' occurs f_1 times, '2' occurs f_2 times, etc. Here the numbers $f_1, \ldots, f_m \geq 0$ and $f_1 + 2.f_2 + \cdots + m.f_m = m$. Note that in this case $l(I) = f_1 + \cdots f_m$; $h(I) = max\{i | f_i > 0\}$.

Given a fixed m we will define $p(m)$ to be the number of different partitions of m. It is still an open problem to find an elementary expression for $p(m)$.

We will also require ordered partitions, $I = (i_1, \ldots, i_r)$, where this time $i_1, \ldots, i_r$ are non-negative integers, and partitions consisting of the same numbers but in a different order are no longer considered equal. We can add ordered partitions of the same length together:

$$I + J = (i_1 + j_1, \ldots, i_r + j_r)$$

So $l(I + J) = r = l(I) = l(J)$, and $(I + J) \vdash (m + n)$.

To avoid confusion over notation we will refer to ordered partitions explicitly whenever we use them.

1.3 Chern Numbers

In this section we shall give a potted account of chern classes and chern numbers and some of their properties. Much of this is lifted straight from [10] where proofs and more details can be found.

The normal chern classes of a manifold, M^n, with a complex normal bundle are elements in the cohomology groups $H^{2i}(M;\mathbb{Z})$. There is one chern class in each group, call it $c_i(M) \in H^{2i}(M;\mathbb{Z})$; $0 \leq i \leq n$, where $c_0(M) = 1$, $\forall M \in MU(2n)$. We shall define the total chern class to be the formal sum:

$$c(M) := c_0(M) + c_1(M) + \cdots + c_n(M).$$

Note that we can define the element:

$$c_{(i_1,\ldots,i_r)} := c_{i_1} \cup \cdots \cup c_{i_r}$$

where '$\cup$' is the cup-product in cohomology; $c_{(i_1,\ldots,i_r)} = c_{\pi(i_1,\ldots,i_r)} \in H^{2(i_1+\cdots+i_r)}(M;\mathbb{Z})$ for any permutation π; $c_{(i_1,\ldots,i_r)} = 0$ for all $i_1 + \cdots + i_r > n$. Now, for any cohomology element $p \in H^{2n}(M^n;\mathbb{Z})$ we can form the Kronecker product $\langle p, \mu_M \rangle \in Z$, where μ_M is the fundamental homology element. In particular we can define:

$$c_I[M^n] := \langle c_{(i_1,\ldots,i_r)}, \mu_M \rangle \in \mathbb{Z}$$

when I corresponds to the partition $i_1 + \cdots + i_r = n$. We shall call $c_I[M^n]$ the chern number of M corresponding to the partition I.

In particular, note the distinction between the Ith chern <u>class</u> $c_I(M^n) \in H^{2n}(M^n; \mathbb{Z})$ and the Ith chern <u>number</u> $c_I[M^n] \in \mathbb{Z}$.

1.4 Initial Results

In this section we will describe some straightforward properties of chern classes and chern numbers, most of which can be found in any suitable text.

We will be interested in the disjoint union of two manifolds, [10] gives us:

Lemma 1.4.1 $c(M\dot{\cup}N) = c(M) + c(N)$

We will also be interested in chern numbers of cross-products, again [10] gives us:

Lemma 1.4.2 $c(M^m \times N^n) = c(M)c(N)$, *from which we can conclude:*

$c_k(M^m \times N^n) = \sum_{i=0}^{k} c_i(M)c_{k-i}(N)$

We can now continue to say that:

Lemma 1.4.3 $c_I(M^m \times N^n) = \sum c_{I_1}(M^m)c_{I_2}(N^n)$,

where the summation is over all ordered partitions I_1, I_2 that sum to I.

<u>**Proof**</u>

$$\begin{aligned} c_I(M \times N) &= \prod_{k=1}^{r}\{\sum_{j=0}^{i_k} c_j(M)c_{i_k-j}(N)\} \\ &= \sum c_{j_1}(M)c_{i_1-j_1}(N)\cdots c_{j_r}(M)c_{i_r-j_r}(N) \end{aligned}$$

summed over every $0 \leq j_k \leq i_k, \forall 1 \leq k \leq r$, but also that $\sum_{k=1}^{r} j_k = m$ and $\sum_{k=1}^{r}(i_k - j_k) = n$, where $l(I) = r$.

But this is exactly the right hand side ||

Notation 1.4.4 *We shall use the notation $M^J := M^{j_1} \times \cdots \times M^{j_s}$ where J is the partition $(j_1, \ldots, j_s)$*

Then it is a straightforward generalization to see that:

Lemma 1.4.5 $c_I[M^J] = \sum c_{I_1}[M^{j_1}] \cdots c_{I_s}[M^{j_s}]$

where J is the partition $(j_1 + \cdots + j_s = n)$ and the summation is over all ordered partitions $I_1 + \cdots + I_s = I$, $I_k \vdash j_k$

We shall now say exactly what the chern numbers of the projective spaces P^n are. According to [10] we have the following:

Lemma 1.4.6 $c(P^n) = (1+x)^{-(n+1)}/x^{n+1} \equiv 0$

and moreover, $\langle x^n, \mu_{P^n} \rangle = 1$

Where $(1+x)^{-(n+1)}/x^{n+1} \equiv 0$ is the (finite) polynomial given by setting $x^{n+1} = 0$ in the (infinite) polynomial $(1+x)^{-(n+1)}$.

As a direct consequence we have that $c_k(P^n) = \binom{-(n+1)}{k} x^k = (-1)^k \binom{n+k}{k} x^k$ and we can see that:

Corollary 1.4.7 $c_I(P^n) = (-1)^n \binom{n+i_1}{i_1} \cdots \binom{n+i_r}{i_r}$

where I is the partition $(i_1 + \cdots i_r = n)$.

We will also be interested in certain algebraic hypersurfaces. In ([10], problem 16E) we consider $H^n_{r,s}$, an n-dimensional, non-singular hypersurface of degree $(1,1)$ in the product $P^r \times P^s$, with $r, s \geq 2$, $n = r + s - 1$. This problem, along with the preceding one, leads us to an evaluation of the total chern class of this space as follows:

Lemma 1.4.8 *The cohomology of $H^n_{r,s} \subseteq P^r \times P^s$ contains the polynomial algebra over $\mathbb{Z}$ with generators x and y, with the restrictions that $x^{r+1} = y^{s+1} = x^r y^s = 0$.*

Proof

Let μ_H and $\mu_{P^r \times P^s}$ be the fundamental homology classes of $H_{r,s}$ and $P^r \times P^s$ respectively.

First of all we shall note that the cohomology of $P^r \times P^s$ is the polynomial algebra over $\mathbb{Z}$ with generators x and y, with the restrictions that $x^{r+1} = y^{s+1} = 0$.

Now, $H_{r,s}$ is complex analytically embedded in $P^r \times P^s$, we wish to find the dual cohomology class u.

Suppose $i : H_{r,s} \hookrightarrow P^r \times P^s$,.

$H_{r,s}$ is of degree $(1,1) \Rightarrow i_*\mu_H = generator + generator \in H_{2n}(P^r \times P^s) \cong \mathbb{Z} \times \mathbb{Z}$.

$H^{2n}(P^r \times P^s)$ is generated by $x^r y^{s-1}$ and $x^{r-1}y^s \Rightarrow \langle x^r y^{s-1}, i_*\mu_H \rangle = 1 = \langle x^{r-1}y^s, i_*\mu_H \rangle$.

But $u \cap \mu_{P^r \times P^s} = i_*\mu_H$, by [10] problem 11C.

$$
\begin{aligned}
\text{So, } 1 &= \langle x^r y^{s-1}, i_*\mu_H \rangle \\
&= \langle x^r y^{s-1}, u \cap \mu_{P^r \times P^s} \rangle \\
&= \langle x^r y^{s-1} \cup u, \mu_{P^r \times P^s} \rangle
\end{aligned}
$$

$\Rightarrow u = kx + y$, where $k \in \mathbb{Z}$.

Similarly $u = x + ly$, where $l \in \mathbb{Z}$.

$\Rightarrow u = x + y$.

Now, we have that for any $z \in H^{2n}(P^r \times P^s; \mathbb{Z})$, $\langle i^* z, \mu_H \rangle = \langle z \cup u, \mu_{P^r \times P^s} \rangle$.

Hence we can conclude that the cohomology of $H_{r,s}$ is the polynomial algebra over $\mathbb{Z}$ with generators x and y, with the restrictions that $x^{r+1} = y^{s+1} = x^r y^s = 0$. ||

Lemma 1.4.9 $c(H_{r,s}) = (1+x)^{-(r+1)}(1+y)^{-(s+1)}(1+x+y)/x^{r+1} \equiv y^{s+1} \equiv x^r y^s \equiv 0$.

Proof

We have that:

$$\begin{aligned} c(H_{r,s}) &= i^*\{c(P^r \times P^s).(1+u)\} \in H^*(H_{r,s};\mathbb{Z}), \text{ by [10], problem 16D} \\ &= i^*\{(1+x)^{-(r+1)}(1+y)^{-(s+1)}(1+x+y)\} \in H^*(H_{r,s};\mathbb{Z}) \end{aligned}$$

This (up to a slight abuse of notation) proves the result. ||

Remarks

1) We can see that $\langle x^r y^{s-1}, \mu_H\rangle = 1 = \langle x^{r-1} y^s, \mu_H\rangle$, so we can note that $c_I[H_{r,s}] =$ coefficient of $x^r y^{s-1}+$ coefficient of $x^{r-1}y^s$ in the polynomial $c_I(H_{r,s})$.

2) We shall also on occasion use the notation H_r^n for a non-singular hypersurface, where $s = n - r + 1$ is assumed.

We can now present $c(H_{r,s})$ more explicitly using the following:

Theorem 1.4.10

$$c(H_{r,s}) = \sum_{u,v} (-1)^{u+v} \binom{r+u}{r} \binom{s+v}{s} \frac{rs - uv}{(r+u)(s+v)} x^u y^v$$

Proof

Put $A = (1+x+y)$, $B = (1+x)^{-(r+1)}$, $C = (1+y)^{-(s+1)}$.

Put $\frac{d^u}{dx^u}\cdot\frac{d^v}{dy^v}A = A_{u,v}$ etc.

Then:

$$(ABC)_{u,v} = \sum_{a+b+c=u}\ \sum_{\alpha+\beta+\gamma=v} \frac{u!v!}{a!b!c!\alpha!\beta!\gamma!} A_{a,\alpha}B_{b,\beta}C_{c,\gamma},$$

by a double usage of Leibniz' trinomial formula.

But, $A_{2,0} = A_{1,1} = A_{0,2} = B_{0,1} = C_{1,0} = 0$

So we have:

$(ABC)_{u,v} = 1.A_{0,0}B_{u,0}C_{0,v} + u.A_{1,0}B_{u-1,0}C_{0,v} + v.A_{0,1}B_{u,0}C_{0,v-1}$

Evaluate at $(0,0) : A_{0,0} = A_{1,0} = A_{0,1} = 1$, $B_{u,0} = (-1)^u\frac{(r+u)!}{r!}$, $C_{0,v} = (-1)^v\frac{(s+v)!}{s!}$.

Hence, the coefficient of x^uy^v in ABC is:

$$\begin{aligned}
& (-1)^{u+v}\left\{\tfrac{(r+u)!(s+v)!}{r!u!s!v!} - \tfrac{u(r+u-1)!(s+v)!}{r!u!s!v!} - \tfrac{v(r+u)!(s+v-1)!}{r!u!s!v!}\right\}\\
= \ & (-1)^{u+v}\tfrac{(r+u-1)!(s+v-1)!}{r!u!s!v!}(rs-uv)\\
= \ & (-1)^{u+v}\binom{r+u}{r}\binom{s+v}{s}\tfrac{rs-uv}{(r+u)(s+v)} \quad \|
\end{aligned}$$

Chapter 2

2.1 Preliminaries

In this chapter we will examine which power of 2 is contained in the highest common factor over all manifolds up to cobordism of certain chern numbers. Due to the results concerning odd primes in Chapter 3 this will, in most cases, actually be the h.c.f.

We will first develop a suitable notation.

Notation 2.1.1 *Let $\alpha(n)$ be the number of 1's in the dyadic expansion of n.*

Let $\nu_2(n) = t$ where $2^t | n$, but $2^{t+1} \not| n$.

Then the following properties are well known:

i) $\nu_2(mn) = \nu_2(m) + \nu_2(n)$,

ii) $\nu_2(m + n) = min^*\{\nu_2(m), \nu_2(n)\}$,

iii) $\nu_2(m) = \alpha(m - 1) - \alpha(m) + 1$,

iv) $\nu_2(m!) = m - \alpha(m)$,

v) $\nu_2\binom{m+n}{n} = \alpha(m) + \alpha(n) - \alpha(m+n)$,

vi) $\alpha(2m) = \alpha(m)$,

vii) $\alpha(2m+1) = \alpha(2m) + 1$,

viii) $\alpha(m) + \alpha(n) \geq \alpha(m+n)$,

Here the notation $A = min^*\{a_1, \ldots, a_k\}$ means:

$$\begin{cases} A = min\{a_1, \ldots, a_k\} & \text{if a strict minimum exists} \\ A > min\{a_1, \ldots, a_k\} & \text{otherwise} \end{cases}$$

In fact we can say slightly more than viii) in special cases:

Result 2.1.2 *If $n = 2^t.\xi$ where ξ is odd, $\xi \geq 3$.*

then $\alpha(2^t) + \alpha(n - 2^t) = \alpha(n)$

Proof

$$\begin{aligned} \alpha(2^t) + \alpha(n-2^t) &= 1 + \alpha\{2^t(\xi - 1)\} \\ &= 1 + \alpha(\xi - 1) \\ &= 1 + \alpha(\xi) - 1 \\ &= \alpha(2^t.\xi) = \alpha(n) \end{aligned}$$

Result 2.1.3 *If $\nu_2(n) = t$ and $r < n$ is odd,*

then $\alpha(r) + \alpha(n-r) \geq \alpha(n) + t$.

Proof

$$\begin{aligned}\alpha(r)+\alpha(n-r) &= \alpha(r-1)+1+\alpha(n-r)\\ &\geq \alpha(n-1)+1\\ &= \alpha(n)-1+t+1\\ &= \alpha(n)+t\end{aligned}$$

Following [12] we will also introduce the following notation.

Notation 2.1.4 *Let* $\rho_r(n) = min\{k|\alpha(n+k) \leq 2k+r\}$

Then, in [12] the following theorem is proved:

Theorem 2.1.5 (Rees and Thomas) $\nu_2\{hcf(c_r c_{n-r}[M^n]|M \in MU(2n))\} = \rho_r(n)$.

The work of this chapter verifies the above formula directly for certain cases and gives a similar result for the chern number $c_1^2 c_{n-2}$. We shall also introduce manifolds $M^n(r)$ such that $\nu_2\{c_r c_{n-r}[M^n(r)]\} = \rho_r(n)$. We do this by examining the Milnor–Novikov generators, $\{K^1, K^2, K^3, \ldots\}$, of the complex cobordism ring. In ([10] problem 16E) the manifolds K^n are described as follows:

Theorem 2.1.6 (Milnor and Novikov) *The ring of all cobordism classes of manifolds with a complex stucture on the stable tangent bundle* $\tau \oplus \epsilon^k$ *is a polynomial algebra over* Z *with generators the manifolds* $K^1, K^2, K^3, \ldots$, *where the manifolds* K^n *are given by:*

$(n+1)$ is prime

$K^n = P^n$.

$(n+1) = p^{t+1}$, where p is prime and $t \geq 1$.

Then $K^n = a_0 P^n + a_1 H^n_{p^t}$, where $a_0(n+1) + a_1\binom{n+1}{p^t} = p$, by the Euclidean algorithm.

$(n+1) = p_1^{t_1} \cdots p_k^{t_k}$, where $p_1, \ldots, p_k$ are distinct primes; $t_1, \ldots, t_k \geq 1$, $k \geq 2$.

Then $K^n = a_0 P^n + \sum_{i=1}^k a_i H^n_{p_i^{t_i}}$,

where $a_0(n+1) + \sum_{i=1}^k a_i\binom{n+1}{p_i^{t_i}} = 1$, by the Euclidean algorithm.

That $hcf\{n+1 = p^{t+1}, \binom{p^{t+1}}{p^t}\} = p$ and $hcf\{n+1 = p_1^{t_1} \cdots p_k^{t_k}, \binom{n+1}{p_1^{t_1}}, \ldots, \binom{n+1}{p_k^{t_k}}\} = 1$ can be seen by a simple counting argument.

Our computation of the chern numbers of the manifolds $K^1, K^2, K^3, \ldots$ will be based on the following observation:

Each K^n is the disjoint union of multiples of a projective space and certain non-singular hypersurfaces. We know the evaluation of chern numbers is distributive over disjoint union so we can use Notation 2.1.1, ii) to say that:

$$\nu_2\{c_I[K^n]\} = min^*\{\nu_2(c_I[P^n]), \nu_2(c_I[hypersurfaces])\} \tag{2.1}$$

In fact we can use the following lemmas to say slightly more.

Lemma 2.1.7 *Suppose $(m,n) = 1$, then by the Euclidean algorithm $\exists a, b \in \mathbb{Z}$ such that $am + bn = 1$.*

Suppose m is even, then b must be odd.

This is obvious, since $(m, b) = 1$, also.

Lemma 2.1.8 *Suppose $(m, n) = 1$, then by the Euclidean algorithm $\exists a, b \in \mathbb{Z}$ such that $am + bn = 1$.*

Suppose m is odd, then solutions exist with b even and with b odd.

If b is even then a must be odd.

Proof

$(m, n) = 1 \Rightarrow \{n, 2n, \ldots, (m-1)n\} = \pi\{1, 2, \ldots, m-1\}$ mod m, for some permutation π.

Hence:

i) $\exists k, 0 < k < m$, such that $kn \equiv 1(m) \Rightarrow \exists \kappa \in \mathbb{Z}$ such that $\kappa m + kn = 1$.

ii) $\exists l, 0 < l < m$, such that $ln \equiv -1(m) \Rightarrow \exists \lambda \in \mathbb{Z}$ such that $\lambda m + (-l)n = 1$.

But, $kn + ln = (k + l)n \equiv 0(m) \Rightarrow k + l = m$, because $0 < k, l < m$.

Now, m is odd $\Rightarrow$ one of k, l is even and the other is odd.

So we have the required solutions a, b.

The last part follows from Lemma 2.1.7||

Lemma 2.1.9 *Suppose $(m_1, \ldots, m_t) = 1$, then by the Euclidean algorithm $\exists a_1, \ldots, a_t \in \mathbb{Z}$, such that $a_1 m_1 + \cdots + a_t m_t = 1$.*

Suppose m_s is odd for some $1 \leq s \leq t$, then we can choose a_s to be odd and the coefficients of any other odd m_i's to be even.

Proof

WLOG let $m_s = m_t$

If $m_1, \ldots, m_{t-1}$ are even then a_s must be odd, obviously.

If one of $m_1, \ldots, m_{t-1}$ is odd then $hcf\{m_1, \ldots, m_{t-1}\}$ is odd and $(hcf\{m_1, \ldots, m_{t-1}\}, m_t) = 1 \Rightarrow$ we can choose a_s to be odd and the other coefficients to be even by Lemma 2.1.8||

These three lemmas tell us that if we do not have a strict minimum in 2.1 we can choose the cö-efficients $a_0, \ldots, a_k$ in the construction of K^n so that exactly one of the appropriate a_i's is odd. This means that we have:

$$\nu_2\{c_I[K^n]\} = min\{\nu_2(c_I[P^n]), \nu_2(c_I[hypersurfaces])\} \tag{2.2}$$

This being the case we can now calculate some chern numbers of the manifolds $K^1, K^2, K^3, \ldots$ explicitly.

2.2 Calculating $\nu_2\{c_n[K^n]\}$

We are now going to evaluate $\nu_2\{c_n[K^n]\}$. In order to do this we will first evaluate $\nu_2\{c_n[P^n]\}$ and $\nu_2\{c_n[H_{r,s}]\}$. Having done this we can then examine the cases individually as the prime decomposition of $(n+1)$ varies.

Lemma 2.2.1 $\nu_2\{c_n[P^n]\} = \alpha(n)$

Proof

We know from Lemma 1.4.7 that $c_n[P^n] = \binom{-(n+1)}{n} = (-1)^n\binom{2n}{n}$

Hence,

$$\begin{aligned}
\nu_2\{c_n[P^n]\} &= \nu_2\binom{2n}{n} \\
&= \alpha(n) + \alpha(n) - \alpha(2n) \\
&= \alpha(n) \|
\end{aligned}$$

Lemma 2.2.2 $c_n[H_{r,s}] = (-1)^n \frac{2n}{rs}\binom{2r-2}{r-1}\binom{2s-2}{s-1}$

Proof

From Lemma 1.4.10:

$$\begin{aligned}
c_n(H_{r,s}) &= (-1)^{r+s-1}\{\binom{2r}{r}\binom{2s-1}{s}\tfrac{rs-rs+r}{(2r)(2s-1)}x^r y^{s-1} \\
&\quad + \binom{2r-1}{r}\binom{2s}{s}\tfrac{rs-rs+s}{(2r-1)(2s)}x^{r-1}y^s\} \\
&= (-1)^n\{x^r y^{s-1}\tfrac{(2r-1)!(2s-2)!}{r!r!s!(s-1)!}r + x^{r-1}y^s\tfrac{(2r-2)!(2s-1)!}{r!(r-1)!s!s!}s\} \\
\Rightarrow c_n[H^n_{r,s}] &= (-1)^n\tfrac{(2r-2)!(2s-2)!}{r!(r-1)!s!(s-1)!}\{(2r-1)+(2s-1)\} \\
&= (-1)^n\tfrac{2n}{rs}\binom{2r-2}{r-1}\binom{2s-2}{s-1} \|
\end{aligned}$$

Lemma 2.2.3 $\nu_2\{c_n[H_{r,s}]\} = \nu_2(n) + \alpha(r) + \alpha(s) - 1$

<u>**Proof**</u>

$$\begin{aligned}
\nu_2\{\tfrac{2n}{rs}\binom{2r-2}{r-1}\binom{2s-2}{s-1}\} &= 1 + \nu_2(n) - \nu_2(r) - \nu_2(s) + \alpha(r-1) + \alpha(s-1) \\
&= 1 + \nu_2(n) - (\alpha(r-1) - \alpha(r) + 1) \\
&\quad -(\alpha(s-1) - \alpha(s) + 1) + \alpha(r-1) + \alpha(s-1) \\
&= \nu_2(n) + \alpha(r) + \alpha(s) - 1 \|
\end{aligned}$$

We know from theorem 2.1.6 that the precise manifolds that are required to make up each K^n depend on the prime decomposition of $(n+1)$. Bearing this in mind we shall now examine more closely $\nu_2\{c_n[H^n_{r,s}]\}$ for the particular hypersurfaces that we shall require.

Result 2.2.4 *Let* $(n+1) = 2^{t+1}$, *where* $t \geq 1$.

Let $r = 2^t = s$.

Then:

$$\begin{aligned}
\nu_2\{c_n[H_{r,s}]\} &= \nu_2(n) + \alpha(2^t) + \alpha(2^t) - 1 \\
&= 1
\end{aligned}$$

Result 2.2.5 *Let* $(n+1) = 2^t.\xi$, *where* $t \geq 1$; $\xi \geq 3$ *is odd.*

Let $r = 2^t$ *and* $s = n + 1 - 2^t$.

Then:

$$\begin{aligned}
\nu_2\{c_n[H_{r,s}]\} &= \nu_2(n) + \alpha(2^t) + \alpha(n+1-2^t) - 1 \\
&= \alpha(n+1) - 1, \text{ by Lemma 2.1.2}
\end{aligned}$$

Result 2.2.6 *Let* $(n+1) = 2^t.\xi.\eta$, *where* $t \geq 1$ *and* $\xi \geq 3, \eta \geq 1$ *are both odd.*

Let $r = \xi$ *and* $s = n + 1 - \xi$.

Then:

$$\begin{aligned} \nu_2\{c_n[H_{r,s}]\} &= \nu_2(n) + \alpha(\xi) + \alpha(n + 1 - \xi) - 1 \\ &\geq \alpha(n + 1) + t - 1, \text{ by Lemma 2.1.3} \end{aligned}$$

Result 2.2.7 *Let* $(n + 1) = \xi.\eta$, *where* $\xi \geq 3, \eta \geq 3$ *are both odd.*

Let $r = \xi$ *and* $s = n + 1 - \xi$.

Then:

$$\begin{aligned} \nu_2\{c_n[H_{r,s}]\} &= \nu_2(n) + \alpha(\xi) + \alpha(n + 1 - \xi) - 1 \\ &\geq 1 + \alpha(n + 1) - 1 \\ &= \alpha(n + 1) \end{aligned}$$

We now come to examine the manifolds K^n, we shall examine the separate cases individually.

Case 2.2.8 *Let* $(n + 1) = p$, *where* p *is prime.*

$K^n = P^n$

$$\nu_2\{c_n[K^n]\} = \alpha(n) = \begin{cases} 1 & \text{if } (n + 1) = 2 \\ \alpha(n + 1) - 1 & \text{if } (n + 1) \text{ is an odd prime} \end{cases}$$

Case 2.2.9 *Let* $(n + 1) = 2^{t+1}$, *where* $t \geq 1$.

$K^n = a_o P^n + a_1 H^n_{2^t}$, *where* $a_0, a_1 \in \mathbb{Z}$.

$\nu_2\{c_n[P^n]\} = t + 1$

$\nu_2\{c_n[H^n_{2^t}]\} = 1$

So, $\nu_2\{c_n[K^n]\} = 1$

Case 2.2.10 *Let* $(n+1) = p^{t+1}$, *where* p *is an odd prime;* $t \geq 1$.

$K^n = a_o P^n + a_1 H^n_{p^t}$, *where* $a_0, a_1 \in \mathbb{Z}$.

$\nu_2\{c_n[P^n]\} = \alpha(n+1) - 1$

$\nu_2\{c_n[H^n_{p^t}]\} \geq \alpha(n+1)$

So, $\nu_2\{c_n[K^n]\} = \alpha(n+1) - 1$

Case 2.2.11 *Let* $(n+1) = 2^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, *where* $t_1, \ldots, t_k \geq 1$; $k \geq 2$; $p_2, \ldots, p_k$ *are distinct, odd primes.*

$$K^n = a_o P^n + a_1 H^n_{2^{t_1}} + \sum_{i=2}^{k} a_i H^n_{p_i^{t_i}}.$$

where $a_0, \ldots, a_k \in \mathbb{Z}$.

$\nu_2\{c_n[P^n]\} = \alpha(n) \geq \alpha(n+1) - 1$

$\nu_2\{c_n[H^n_{2^t}]\} = \alpha(n+1) - 1$

$\nu_2\{c_n[H^n_{p_i^{t_i}}]\} \geq \alpha(n) \geq \alpha(n+1) - 1$

So, $\nu_2\{c_n[K^n]\} = \alpha(n+1) - 1$

Case 2.2.12 *Let* $(n+1) = p_1^{t_1} \cdots p_k^{t_k}$, *where* $t_1, \ldots, t_k \geq 1$; $k \geq 2$; $p_1, \ldots, p_k$ *are distinct, odd primes.*

$$K^n = a_0 P^n + \sum_{i=1}^{k} a_i H^n_{p_i^{t_i}}.$$

where $a_0, \ldots, a_k \in \mathbb{Z}$.

$\nu_2\{c_n[P^n]\} = \alpha(n+1) - 1$

$\nu_2\{c_n[H^n_{p_i^{t_i}}]\} \geq \alpha(n+1)$

So, $\nu_2\{c_n[K^n]\} = \alpha(n+1) - 1$

So we can summarize the above in the following:

Proposition 2.2.13 $\nu_2\{c_n[K^n]\} = \begin{cases} 1 & \textit{if } \alpha(n+1) = 1 \\ \alpha(n+1) - 1 & \textit{if } \alpha(n+1) \geq 2 \end{cases}$

We would now like to show that:

$$min\{\nu_2\{c_n[M^n]\} | M^n \in MU(2n)\} = \rho_0(n). \tag{2.3}$$

Now, we can assume M^n is connected because of 1.4 and so, from 2.1.6, M^n must be a cross-product of manifolds from $\{K^1, K^2, K^3, \ldots\}$.

We will prove equation 2.3 in two stages. First we will prove the following:

Lemma 2.2.14 $\nu_2\{c_n[M^n]\} \geq \rho_0(n), \ \forall M^n \in MU(2n)$

Proof

We will use induction.

The result is obviously true for $M^n = K^n$.

Suppose $M^n = U^u \times V^v$ and the result is true for U and V.

Then:

$$
\begin{aligned}
\nu_2\{c_n[U \times V]\} &= \nu_2\{c_u[U]c_v[V]\}, \text{ by lemma 1.4.5} \\
&= \nu_2\{c_u[U]\} + \nu_2\{c_v[V]\} \\
&\geq \kappa_1 + \kappa_2
\end{aligned}
$$

$$
\text{where } \begin{cases} \kappa_1 = min\{k|\alpha(u+k) \leq 2k\} \\ \kappa_2 = min\{k|\alpha(v+k) \leq 2k\} \end{cases}
$$

$$
\begin{aligned}
But,\ 2(\kappa_1 + \kappa_2) &= 2\kappa_1 + 2\kappa_2 \\
&\geq \alpha(u+\kappa_1) + \alpha(v+\kappa_2) \\
&\geq \alpha(u+v+\kappa_1+\kappa_2) \\
&= \alpha(n+\kappa_1+\kappa_2)
\end{aligned}
$$

$\Rightarrow (\kappa_1 + \kappa_2) \geq min\{k|\alpha(n+k) \leq 2k\} = \rho_0(n)||$

To prove equation 2.3 it will now suffice to show that $\exists J \vdash n\ s.t.\ \nu_2\{c_n[K^J]\} = \rho_0(n)$. This will be dealt with in the framework of a more general result in Section 2.6.

2.3 Calculating $\nu_2\{c_1c_{n-1}[K^n]\}$

We are now going to evaluate $\nu_2\{c_1c_{n-1}[K^n]\}$. In order to do this we will first evaluate $\nu_2\{c_1c_{n-1}[P^n]\}$ and $\nu_2\{c_1c_{n-1}[H^n_{r,s}]\}$. Having done this we can then examine the cases individually as the prime decomposition of $n+1$ varies.

Lemma 2.3.1 $\nu_2\{c_1c_{n-1}[P^n]\} = 2\alpha(n) - \alpha(n+1)$

Proof

By 1.4.7,

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[P^n]\} &= \nu_2\{\binom{n+1}{1}\binom{2n-1}{n-1}\} \\
&= \nu_2(n+1) + \nu_2\binom{2n-1}{n-1} \\
&= \alpha(n) - \alpha(n+1) + 1 + \alpha(n-1) + \alpha(n) - \alpha(2n-1) \\
&= 2\alpha(n) - \alpha(n+1) + 1 + \alpha(n-1) - \alpha(n-1) - 1 \\
&= 2\alpha(n) - \alpha(n+1) \|
\end{aligned}$$

Lemma 2.3.2 $c_1c_{n-1}[H^n_{r,s}] = (-1)^n\binom{2r-2}{r}\binom{2s-2}{s}\frac{r^2+6rs+s^2-2r-2s}{(r-1)(s-1)}$

Proof

We require to find the coefficients of x^ry^{s-1} and $x^{r-1}y^s$ in $c_1c_{r+s-2}(H_{r,s})$

$c_1(H_{r,s}) = -rx - sy$

From Lemma 1.4.10, $c_{r+s-2}(H_{r,s}) = (-1)^{r+s-2}\{Ax^{r-2}y^s + Bx^{r-1}y^{s-1}$

$+ Cx^ry^{s-2}\}$,

Where:

$A = \frac{(2r-3)!(2s)!}{(r-2)!r!s!s!}$

$C = \frac{(2r)!(2s-3)!}{r!r!(s-2)!s!}$

$B = \frac{(2r-2)!(2s-2)!}{(r-1)!r!(s-1)!s!}(r+s-1)$

We can now put all of the above together to say that:

$$\begin{aligned} c_1c_{r+s-2}(H^{r,s}) &= (-1)^{r+s-1}(Ax^{r-2}y^s + Bx^{r-1}y^{s-1} + Cx^ry^{s-2})(rx+sy) \\ &= (-1)^{r+s-1}(Ar+Bs)x^{r-1}y^s + (Br+Cs)x^ry^{s-1} \end{aligned}$$

Evaluating this on the fundamental homology class we can say that:

$$\begin{aligned} c_1c_{r+s-2}[H^{r,s}] &= (-1)^{r+s-1}\{Ar + B(r+s) + Cs\} \\ &= (-1)^{r+s-1}\{\tfrac{(2r-3)!(2s)!}{(r-2)!r!s!s!}r + \tfrac{(2r-2)!(2s-2)!}{(r-1)!r!(s-1)!s!}(r+s-1)(r+s) + \tfrac{(2r)!(2s-3)!}{r!r!(s-2)!s!}s\} \\ &= (-1)^{r+s-1}\tfrac{(2r-3)!(2s-3)!}{r!r!s!s!}\{2rs(2s-1)(2s-2)r(r-1) \\ &\quad +(2r-2)(2s-2)rs(r+s)(r+s-1) + 2rs(2r-1)(2r-2)s(s-1)\} \\ &= (-1)^{r+s-1}\tfrac{(2r-3)!(2s-3)!}{r!r!s!s!}\{4r(r-1)s(s-1)(r^2+6rs+s^2-2r-2s)\} \\ &= (-1)^{r+s-1}\tfrac{4(2r-3)!(2s-3)!}{r!(r-2)!s!(s-2)!}(r^2+6rs+s^2-2r-2s) \\ &= (-1)^{r+s-1}\tfrac{(2r-2)(2s-2)}{(r-1)(s-1)}\tfrac{(2r-3)!(2s-3)!}{r!(r-2)!s!\,(s-2)!}(r^2+6rs+s^2-2r-2s) \\ &= (-1)^{r+s-1}\binom{2r-2}{r}\binom{2s-2}{s}\tfrac{r^2+6rs+s^2-2r-2s}{(r-1)(s-1)} \| \end{aligned}$$

Lemma 2.3.3 $\nu_2\{c_1c_{n-1}[H_{r,s}]\} = \alpha(r) + \alpha(s) - 2 + \nu_2(r^2+6rs+s^2-2r-2s)$

Proof

$\nu_2\{\binom{2r-2}{r}\binom{2s-2}{s}\frac{r^2+6rs+s^2-2r-2s}{(r-1)(s-1)}\} =:$

$\alpha(r) + \alpha(r-2) - \alpha(2r-2) + \alpha(s) + \alpha(s-2) - \alpha(2s-2) + \nu_2(r^2+6rs+s^2-$

$2r-2s) - (\alpha(r-2) - \alpha(r-1) + 1) - (\alpha(s-2) - \alpha(s-1) + 1)$

$= \alpha(r) + \alpha(s) - 2 + \nu_2(r^2+6rs+s^2-2r-2s)\|$

We now require to examine $\nu_2\{c_1c_{n-1}[H_{r,s}]\}$ for the particular hypersurfaces that we require to make up the Milnor-Novikov generators, K^n. As before we shall examine each case individually.

Notation 2.3.4 *We shall use the notation 'e' for an even number and 'o' for an odd number, noting that:*

$$\nu_2\{e + \cdots + e + \overbrace{o + \cdots + o}^{\text{odd number}}\} = 0$$

Result 2.3.5 *Let* $n + 1 = 2^{t+1}$, *where* $t \geq 1$

Let $r = s = 2^t$

Then:

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[H_{r,s}]\} &= 1 + 1 - 2 + \nu_2\{2^{2t} + 3.2^{2t+1} + 2^{2t} - 2^{t+2}\} \\
&= \nu_2\{2^{2t+3} - 2^{t+2}\} \\
&= t + 2 \\
&= \alpha(n) + 1 \|
\end{aligned}$$

Result 2.3.6 *Let* $n + 1 = 2.\xi$ *where* ξ *is odd.*

Let $r = 2,\ s = n - 1$

Note that when, as in this case, $4|(n-1)$, *then* $\alpha(n-1) = \alpha(n+1) - 1$.

Then:

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[H_{r,s}]\} &= 1+\alpha(n-1)-2+\nu_2\{4+12.(n-1)+(n-1)^2-4 \\
&\quad -2.(n-1)\} \\
&= \alpha(n+1)-2+\nu_2\{12.(n-1)+(n-1)^2-2.(n-1)\} \\
&\geq \alpha(n+1)+1\|
\end{aligned}$$

Result 2.3.7 *Let $n+1=2^t.\xi$, with ξ odd; $t \geq 2$, $\xi \geqslant 3$.*

Let $r=2^t$ and $s=n+1-2^t$.

Then:

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[H_{r,s}]\} &= \alpha(2^t)+\alpha(n+1-2^t)-2+\nu_2\{2^{2t}+6.2^t(n+1-2^t) \\
&\quad +(n+1-2^t)^2-2^{t+1}-2.(n+1-2^t)\} \\
&= \alpha(n+1)-2+t+1, \textit{ by lemma 2.1.2} \\
&= \alpha(n)\|
\end{aligned}$$

Result 2.3.8 *Let $n+1=2^t.\xi.\eta$; with ξ and η both odd and $t \geq 1$, $\xi \geqslant 3$, $\eta \geqslant 1$.*

Let $r=\xi$ and $s=n+1-\xi$.

Then:

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[H_{r,s}]\} &= \alpha(\xi)+\alpha(n+1-\xi)-2+\nu_2\{\xi^2+6.\xi.(n+1-\xi) \\
&\quad +(n+1-\xi)^2-2.\xi-2.(n+1-\xi)\} \\
&\geq \alpha(n+1)+t-2+\nu_2\{-4.\xi^2+4.\xi.(n+1) \\
&\quad +(n+1)^2-2.(n+1)\}, \textit{ by Lemma 2.1.3} \\
&= \alpha(n)-1+\nu_2\{-4.\xi^2+\xi.4.(n+1)+(n+1)(n-1)\} \\
&= \alpha(n)-1+2 \\
&= \alpha(n)+1\|
\end{aligned}$$

Result 2.3.9 *Let $n+1=\xi.\eta$; with ξ and η both odd.*

Let $r=\xi$ and $s=n+1-\xi$.

Then:

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[H_{r,s}]\} &= \alpha(\xi)+\alpha(n+1-\xi)-2+\nu_2\{\xi^2+6.\xi.(n+1-\xi) \\
&\quad +(n+1-\xi)^2-2.\xi-2.(n+1-\xi)\} \\
&\geq \alpha(n+1)-2+\nu_2\{o+e+e-e-e\} \\
&= \alpha(n)+1-2 \\
&= \alpha(n)-1 \|
\end{aligned}$$

We can now use the above results and Lemmas 2.1.7, 2.1.8 and 2.1.9 to evaluate $\nu_2\{c_1c_{n-1}[K^n]\}$. Again we shall examine the different cases separately.

Case 2.3.10 *Let $n+1=2$;*

Then $K^1=P^1$ and $\nu_2\{c_1c_0[P^1]\}=1$

Case 2.3.11 *Let $n+1=p$, where p is an odd prime.*

Then $K^n=P^n$

$$\begin{aligned}
\nu_2\{c_1c_{n-1}[P^n]\} &= 2.\alpha(n)-\alpha(n+1) \\
&= 2.\alpha(n)-\alpha(n)-1 \\
&= \alpha(n)-1
\end{aligned}$$

Case 2.3.12 *Let $n+1=2^{t+1}$, where $t\geq 1$.*

Then $K^n=a_0P^n+a_1H^n_{2^t}$ for some $a_0,a_1\in\mathbb{Z}$

$$\begin{aligned}\nu_2\{c_1c_{n-1}[P^n]\} &= 2\alpha(n)-\alpha(n+1)\\ &= 2\alpha(n)-1\\ &\geq \alpha(n)+1\|\end{aligned}$$

$\nu_2\{c_1c_{n-1}[H^n_{2^t}]\} = \alpha(n)+1\|$

So, $\nu_2\{c_1c_{n-1}[K^n]\} = \alpha(n)+1$

Case 2.3.13 *Let* $n+1=p^{t+1}$*, where* p *is an odd prime;* $t\geq 1$.

Then $K^n = a_0P^n + a_1H^n_{p^t}$ *for some* $a_0, a_1 \in \mathbb{Z}$.

$$\begin{aligned}\nu_2\{c_1c_{n-1}[P^n]\} &= 2\alpha(n)-\alpha(n+1)\\ &= 2\alpha(n)-\alpha(n)-1\\ &= \alpha(n)-1\|\end{aligned}$$

$\nu_2\{c_1c_{n-1}[H^n_{p^t}]\} \geq \alpha(n)-1$

So, $\nu_2\{c_1c_{n-1}[K^n]\} = \alpha(n)-1$

Case 2.3.14 *Let* $n+1 = 2^{t_1}p_2^{t_2}\cdots p_k^{t_k}$*, where* $p_2,\ldots,p_k$ *are distinct odd primes;* $t_1,\ldots,t_k\geq 1; k\geq 2.$

Then:

$$K^n = a_0P^n + a_1H^n_{2^{t_1}} + \sum_{i=2}^{k} a_iH^n_{p_i^{t_i}}$$

for some $a_0,\ldots,a_k \in \mathbb{Z}$

$$\begin{aligned}\nu_2\{c_1c_{n-1}[P^n]\} &= 2\alpha(n)-\alpha(n+1)\\ &= \alpha(n)+\{\alpha(n)-\alpha(n+1)+1\}-1\\ &= \alpha(n)+\nu_2(n+1)-1\\ &= \alpha(n)+t_1-1\|\end{aligned}$$

$\nu_2\{c_1c_{n-1}[H^n_{p_i^{t_i}}]\} \geq \alpha(n)+1$

$$\nu_2\{c_1c_{n-1}[H^n_{2^{t_1}}]\} = \begin{cases} \geq \alpha(n+1)+1 & t_1 = 1 \\ \alpha(n) & t_1 \geq 2 \end{cases}$$

So, $\nu_2\{c_1c_{n-1}[K^n]\} = \alpha(n)$

Case 2.3.15 *Let* $n+1 = p_1^{t_1}\cdots p_k^{t_k}$, *where* $p_1,\ldots,p_k$ *are distinct, odd primes;* $t_1,\ldots,t_k \geq 1; k \geq 2.$

Then:

$$K^n = a_0P^n + \sum_{i=1}^{k} a_iH^n_{p_i^{t_i}}$$

for some $a_0,\ldots,a_k \in \mathbb{Z}$.

$$\begin{aligned}\nu_2\{c_1c_{n-1}[P^n]\} &= 2\alpha(n)-\alpha(n+1)\\ &= 2\alpha(n)-\alpha(n)-1\\ &= \alpha(n)-1\|\end{aligned}$$

$\nu_2\{c_1c_{n-1}[H^n_{p_i^{t_i}}]\} \geq \alpha(n)-1$

So, $\nu_2\{c_1c_{n-1}[K^n]\} = \alpha(n)-1$

We have now examined every possibility and we can summarize Cases 1 to 5 in the following:

Proposition 2.3.16

$$\nu_2\{c_1c_{n-1}[K^n]\} = \begin{cases} 1 & \textit{if } n+1=2 \\ \alpha(n)-1 & \textit{if } n+1 \textit{ is odd} \\ \alpha(n) & \textit{if } n+1 \textit{ is even but not a power of } 2 \\ \alpha(n)+1 & \textit{if } n+1=2^t,\ t\geq 2 \end{cases}$$

We would now like to show that:

$$min\{\nu_2\{c_1c_{n-1}[M^n]\}|M^n \in MU(2n)\} = \rho_1(n) \tag{2.4}$$

As in Section 2.2 we will first show the following:

Lemma 2.3.17 $\nu_2\{c_1c_{n-1}[M^n]\} \geq \rho_1(n),\ \forall M^n \in MU(2n)$

<u>**Proof**</u>

We will use induction.

The result is true for $M^n = K^n$.

Suppose $M^n = U^u \times V^v$ and the result is true for U and V.

Then, by Lemma 1.4.5 we have:

$$\begin{aligned} \nu_2\{c_1c_{n-1}[U\times V]\} &= \nu_2\{c_u[U]c_1c_{v-1}[V] + c_1c_{u-1}[U]c_v[V]\} \\ &= min^*\{\nu_2\{c_u[U]c_1c_{v-1}[V]\}, \nu_2\{c_1c_{u-1}[U]c_v[V]\}\} \end{aligned}$$

WLOG look at:

$$\begin{aligned}
\nu_2\{c_u[U]c_1c_{v-1}[V]\} &= \nu_2\{c_u[U]\} + \nu_2\{c_1c_{v-1}[V]\} \\
&\geq \kappa_1 + \kappa_2
\end{aligned}$$

where $\begin{cases} \kappa_1 = min\{k|\alpha(u+k) \leq 2k\}, \text{ by lemma 2.2.14} \\ \kappa_2 = min\{k|\alpha(v+k) \leq 2k+1\} \end{cases}$

$$\begin{aligned}
But,\ 2(\kappa_1+\kappa_2)+1 &= 2\kappa_1 + (2\kappa_2+1) \\
&\geq \alpha(u+\kappa_1) + \alpha(v+\kappa_2) \\
&\geq \alpha(u+v+\kappa_1+\kappa_2) \\
&= \alpha(n+\kappa_1+\kappa_2)
\end{aligned}$$

$\Rightarrow \kappa_1 + \kappa_2 \geq min\{k|\alpha(n+k) \leq 2k+1\} \|$

Again, as in Section 2.2, we now require to find $J \vdash n,\ s.t.\ \nu_2\{c_1c_{n-1}[K^J]\} = \rho_1(n)$ in order to prove (2.4) This will be covered in Section 2.6.

2.4 Calculating $\nu_2\{c_2c_{n-2}[K^n]\}$

We are now going to evaluate $\nu_2\{c_2c_{n-2}[K^n]\}$. As in previous cases we begin by working out $\nu_2\{c_2c_{n-2}[P^n]\}$ and $\nu_2\{c_2c_{n-2}[H_{r,s}]\}$.

Lemma 2.4.1

$$\nu_2\{c_2c_{n-2}[P^n]\} = \begin{cases} 2\alpha(n) - \alpha(n+2) & \text{if } n \text{ is even} \\ \alpha(n-2) + \alpha(n) - \alpha(n+2) + 2 & \text{if } n \text{ is odd} \end{cases}$$

Proof

By 1.4.7

$$c_2c_{n-2}[P^n] = (-1)^n \binom{n+2}{2}\binom{2n-2}{n-2}$$

$$\begin{aligned} \Rightarrow \nu_2\{c_2c_{n-2}[P^n]\} &= \nu_2\{\binom{n+2}{2}\binom{2n-2}{n-2}\} \\ &= \alpha(2) + \alpha(n) - \alpha(n+2) + \alpha(n-2) + \alpha(n) \\ &\quad -\alpha(2n-2) \\ &= 1 + 2\alpha(n) + \alpha(n-2) - \alpha(n-1) - \alpha(n+2) \end{aligned}$$

So, for <u>n even</u>:

$$\begin{aligned} \nu_2\{c_2c_{n-2}[P^n]\} &= 1 + 2\alpha(n) - \alpha(n+2) + \alpha(n-1) - 1 - \alpha(n-1) \\ &= 2\alpha(n) - \alpha(n+2) \end{aligned}$$

and for <u>n odd</u>:

$$\begin{aligned} \nu_2\{c_2c_{n-2}[P^n]\} &= \alpha(n-2) + \alpha(n) - \alpha(n+2) + \alpha(n-1) + 1 \\ &\quad -\alpha(n-1) + 1 \\ &= \alpha(n-2) + \alpha(n) - \alpha(n+2) + 2 \end{aligned}$$

Lemma 2.4.2 $c_2c_{n-2}[H^n_{r,s}] = (-1)^n \frac{(2r-4)!(2s-4)!}{r!(r-2)!s!(s-2)!}\{6(r+s)^4 - (r+s)^3(4rs+24) + (r+s)^2(58rs-30) - (r+s)(32r^2s^2 - 2rs - 138) - (16r^2s^2 + 128rs + 72)\}$

<u>**Proof**</u>

From Lemma 1.4.10 we can see that:

$$\begin{aligned} c_2(H_{r,s}) &= \binom{r+2}{r}\binom{s}{s}\frac{rs-0}{(r+2)s}x^2 + \binom{r+1}{r}\binom{s+1}{s}\frac{rs-1}{(r+1)(s+1)}xy + \binom{r}{r}\binom{s+2}{s}\frac{rs-0}{r(s+2)}y^2 \\ &= \tfrac{1}{2}r(r+1)x^2 + (rs-1)xy + \tfrac{1}{2}s(s+1)y^2 \end{aligned}$$

From Lemma 1.4.10 again we have that:

$$c_{r+s-3}(H_{r,s}) = (-1)^{r+s-3}\{Ax^ry^{s-3} + Bx^{r-1}y^{s-2} + Cx^{r-2}y^{r-1} + Dx^{r-3}y^s\}$$

Where:

$A = \frac{(2r-1)!(2s-4)!}{r!r!s!(s-3)!}(rs - rs + 3r) = \frac{3(2r-1)!(2s-4)!}{r!(r-1)!s!(s-3)!}$

$B = \frac{(2r-2)!(2s-3)!}{r!(r-1)!s!(s-2)!}(rs - rs + 2r + s - 2) = \frac{(2r-2)!(2s-3)!}{r!(r-1)!s!(s-2)!}(2r + s - 2)$

By symmetry $C = \frac{(2r-3)!(2s-2)!}{r!(r-2)!s!(s-1)!}(r + 2s - 2)$

$$D = \frac{3(2r-4)!(2s-1)!}{r!(r-3)!s!(s-1)!}$$

Hence,

$$c_2c_{r+s-3}(H_{r,s}) = (-1)^n x^r y^{s-1}\{\tfrac{1}{2}r(r+1)C + (rs-1)B + \tfrac{1}{2}s(s+1)A\}$$
$$+ (-1)^n x^{r-1}y^s\{\tfrac{1}{2}r(r+1)D + (rs-1)C + \tfrac{1}{2}s(s+1)B\}$$

Now, the coefficient of x^ry^{s-1} in this expression is:

$$\tfrac{1}{2}r(r+1)\frac{(2r-3)!(2s-2)!}{r!(r-2)!s!(s-1)!}(r+2s-2) + (rs-1)\frac{(2r-2)!(2s-3)!}{r!(r-1)!s!(s-2)!}(2r+s-2)$$
$$+ \tfrac{1}{2}s(s+1)\frac{3(2r-1)!(2s-4)!}{r!(r-1)!s!(s-3)!}$$

$= \frac{(2r-4)!(2s-4)!}{r!(r-1)!s!(s-1)!}\{\frac{1}{2}r(r+1)(2r-3)(r-1)(2s-2)(2s-3)(r+2s-2)+(rs-1)(2r-2)(2r-3)(2s-3)(s-1)(2r+s-2)+\frac{1}{2}s(s+1)3(2r-1)(2r-2)(2r-3)(s-1)(s-2)\}$

$= \frac{(2r-4)!(2s-4)!}{r!(r-1)!s!(s-1)!}\{4r^4s - 6r^4 + 24r^3s^2 - 46r^3s + 15r^3 + 20r^2s^3 - 68r^2s^2 + 24r^2s + 27r^2 - 36rs^3 + 46rs^2 + 94rs - 78r + 9s^3 + 3s^2 - 60s + 36\}$

By symmetry, the coefficient of $x^{r-1}y^s$ is:

$\frac{(2r-4)!(2s-4)!}{r!(r-1)!s!(s-1)!}\{20r^3s^2 - 36r^3s + 9r^3 + 24r^2s^3 - 68r^2s^2 + 46r^2s + 3r^2 + 4rs^4 - 46rs^3 + 24rs^2 + 94rs - 60r - 6s^4 + 15s^3 + 27s^2 - 78s + 36\}$

Finally we add the coefficients to get:

$c_2c_{n-2}(H^n_{r,s}) = (-1)^n \frac{(2r-4)!(2s-4)!}{r!(r-1)!s!(s-1)!}\xi$

Where $\xi =$:

$-(4r^4s + 44r^3s^2 + 44r^2s^3 + 4rs^4) + (6r^4 + 82r^3s + 136r^2s^2 + 82rs^3 + 6s^4) - (24r^3 + 70r^2s + 70rs^2 + 24s^3) - (30r^2 + 188rs + 30s^2) + (138r + 138s) - 72$

$= (r+s)\{-(4r^3s + 40r^2s^2 + 4rs^3) + (6r^3 + 76r^2s + 76rs^2 + 6s^3) - (24r^2 + 46rs + 24s^2) - (30r + 30s) + 138\} - 72 - 16r^2s^2 - 128rs$

$= -(r+s)^2\{(4r^2s + 4rs^2) - (6r^2 + 70rs + 6s^2) + (24r + 24s) + 30\} - (r+s)\{32r^2s^2 - 2rs - 138\} - \{16r^2s^2 + 128rs + 72\}$

$= -(r+s)^3\{4rs - (6r + 6s) + 24\} - (r+s)^2\{-58rs + 30\} - (r+s)\{32r^2s^2 - 2rs - 138\} - \{16r^2s^2 + 128rs - 72\}$

$= 6(r+s)^4 - (r+s)^3\{4rs + 24\} + (r+s)^2\{58rs - 30\} - (r+s)\{32r^2s^2 - 2rs - 138\} - \{16r^2s^2 + 128rs + 72\}$

This completes the proof.||

Lemma 2.4.3 $\nu_2\{c_2c_{n-2}[H^n_{r,s}]\} = \alpha(r) + \alpha(s) - 4 + \nu_2(\xi)$

<u>Proof</u>

$$\begin{aligned}\nu_2\{\tfrac{(2r-4)!(2s-4)!}{r!(r-1)!s!(s-1)!}\xi\} &= 2r-4-\alpha(2r-4)+2s-4-\alpha(2s-4)+\alpha(r)-r \\ &\quad +\alpha(r-2)-(r-2)+\alpha(s)-s+\alpha(s-2)-(s-2) \\ &\quad +\nu_2(\xi) \\ &= \alpha(r)+\alpha(s)-4+\nu_2(\xi)\|\end{aligned}$$

For ease of computation later on we shall now examine the divisibility of ξ for various values of r and s. We shall use the same notation as in Notation 2.3.4

Case 2.4.4 *Let r be even and s be odd.*

$$\begin{aligned}\xi &= 2.o-2.e+2.o(e-o)-2.o(e-e-o)-2.e \\ &= 2(o-e+e-o-e+e+o-e) \\ &= 2.o\end{aligned}$$

$\Rightarrow \nu_2(\xi) = 1\|$

Case 2.4.5 *Let $r = 2.\rho$ and $s = 2.\sigma$, for some $\rho, \sigma \in \mathbb{Z}$.*

$$\begin{aligned}\xi &= 2^5(\rho+\sigma)^4.o-2^6(\rho+\sigma)^3(2.\rho.\sigma+o)+2^3(\rho+\sigma)^2(4.\rho.\sigma.o-o) \\ &\quad -2^2(\rho+\sigma)(2^8\rho^2\sigma^2-4.\rho.\sigma-o)-2^3(2^5.\rho^2.\sigma^2+2^6.\rho.\sigma+o) \\ &= 2^4.X-2^3(\rho+\sigma)^2.o+2^2(\rho+\sigma).o-2^3.o, \textit{ for some } X \in \mathbb{Z}\end{aligned}$$

$$\textit{So } \nu_2(\xi) = \begin{cases} 2 & \textit{if } \nu_2(r+s) = 1 \\ \geq 4 & \textit{if } \nu_2(r+s) = 2 \\ 3 & \textit{if } \nu_2(r+s) \geq 3 \end{cases}$$

Case 2.4.6 *Let r be odd and s be odd. Let $r+s=2.\tau$ for some $\tau \in \mathbb{Z}$.*

$$\begin{aligned}\xi &= 2^5.\tau^4.o - 2^5.\tau^3(o+e) + 2^3.\tau^2(o-o) - 2^2.\tau(2^4.o - o - o) \\ &\quad -2^3(2.o + 2^4.o + o) \\ &= 2.X + 2^2.\tau(o+o) - 2^3.o, \textit{ for some } X \in \mathbb{Z}\end{aligned}$$

$\Rightarrow \nu_2(\xi) \geq 3.$

We shall now examine more closely $\nu_2\{c_2c_{n-2}[H_{r,s}]\}$ and $\nu_2\{c_2c_{n-2}[P_n]\}$ for the particular manifolds that we shall require.

Result 2.4.7 *Let $(n+1) = 2^{t+1}$, for $t \geq 1$.*

$$\begin{aligned}\nu_2\{c_2c_{n-2}[P^n]\} &= \alpha(2^{t+1}-3) + \alpha(2^{t+1}-1) - \alpha(2^{t+1}+1) + 2 \\ &= t + (t+1) - 2 + 2 \\ &= 2t+1\end{aligned}$$

Result 2.4.8 *Let $(n+1) = 2^{t+1}$, where $t \geq 1$.*

Let $r = 2^t = s$.

Then:

$$\begin{aligned}\nu_2\{c_2c_{n-2}[H_{r,s}]\} &= \alpha(2^t) + \alpha(2^t) - 4 + \nu_2(\xi) \\ &= \begin{cases} 3 & \textit{if } t = 1 \\ 1 & \textit{if } t \geq 2 \end{cases}\end{aligned}$$

Result 2.4.9 *Let $(n+1) = 2^t.\eta$, where $t \geq 1$; $\eta \geq 3$ is odd.*

$$
\begin{aligned}
\nu_2\{c_2c_{n-2}[P^n]\} &= \alpha(2^t.\eta - 3) + \alpha(2^t.\eta - 1) - \alpha(2^t.\eta + 1) + 2 \\
&= \alpha(2^t.\eta - 3) + \alpha(2^t\eta) - 1 + t - \alpha(2^t.\eta) - 1 + 2 \\
&= \alpha(2^t.\eta - 3) + t \\
&= \begin{cases} \geq \alpha(n+1) + t + 1 & if\, t = 1 \\ \alpha(n+1) + 2t - 2 & if\, t \geq 2 \end{cases}
\end{aligned}
$$

Result 2.4.10 *Let $(n+1) = 2^t.\eta$, where $t \geq 1$; $\eta \geq 3$ is odd.*

Let $r = 2^t$ and $s = n + 1 - 2^t$.

Then:

$$
\begin{aligned}
\nu_2\{c_2c_{n-2}[H_{r,s}]\} &= \alpha(2^t) + \alpha(n + 1 - 2^t) - 4 + \nu_2(\xi) \\
&= \alpha(n+1) - 4 + \nu_2(\xi) \\
&= \begin{cases} \alpha(n+1) - 2 & if\, t = 1 \\ \geq \alpha(n+1) & if\, t = 2 \\ \alpha(n+1) - 1 & if\, t \geq 3 \end{cases}
\end{aligned}
$$

Result 2.4.11 *Let $(n+1) = 2^t.\eta.\zeta$, where $t \geq 1$; $\eta \geq 3, \zeta \geq 1$ are both odd.*

Let $r = \eta$ and $s = n + 1 - \eta$.

Then:

$$
\begin{aligned}
\nu_2\{c_2c_{n-2}[H_{r,s}]\} &= \alpha(\eta) + \alpha(n + 1 - \eta) - 4 + \nu_2(\xi) \\
&\geq \alpha(n+1) + t - 4 + 3 \\
&= \alpha(n+1) + t - 1
\end{aligned}
$$

Result 2.4.12 *Let $(n+1) = \eta.\zeta$, where $\eta \geq 3, \zeta \geq 3$ are both odd.*

Let $r = \eta$ and $s = n + 1 - \eta$.

Then:

$$\begin{aligned}\nu_2\{c_2c_{n-2}[H_{r,s}]\} &= \alpha(\eta)+\alpha(n+1-\eta)-4+\nu_2(\xi)\\ &\geq \alpha(n+1)-4+1\\ &= \alpha(n+1)-3\end{aligned}$$

We can now use the above results and Lemmas 2.1.7, 2.1.8, 2.1.9 to examine $\nu_2\{c_2c_{n-2}[K^n]\}$. We shall, as before, examine the different cases separately.

Case 2.4.13 *Let $(n+1)=p$, where p is an odd prime.*

Then $K^n = P^n$.

$$\begin{aligned}\nu_2\{c_2c_{n-2}[K^n]\} &= 2\alpha(n)-\alpha(n+2)\\ &= 2\alpha(n+1)-2-\alpha(n+2)\end{aligned}$$

Case 2.4.14 *Let $(n+1)=2^{t+1}$, where $t\geq 1$.*

Then $K^n = a_0P^n + a_1H^n_{2^t}$, where $a_0, a_1 \in \mathbb{Z}$.

$\nu_2\{c_2c_{n-2}[P^n]\} = 2t+1$

$$\nu_2\{c_2c_{n-2}[H^n_{2^t}]\} = \begin{cases} 3 & \textit{if } t=1\\ 1 & \textit{if } t\geq 2\end{cases}$$

$$\textit{So, } \nu_2\{c_2c_{n-2}[K^n]\} = \begin{cases} 3 & \textit{if } t=1\\ 1 & \textit{if } t\geq 2\end{cases}$$

Case 2.4.15 *Let $(n+1)=p^{t+1}$, where p is an odd prime; $t\geq 1$.*

Then $K^n = a_0P^n + a_1H^n_{p^t}$, where $a_0, a_1 \in \mathbb{Z}$.

$$\begin{aligned}\nu_2\{c_2c_{n-2}[Pn]\} &= 2\alpha(n)-\alpha(n+2)\\ &= 2\alpha(n+1)-2-\alpha(n+2)\end{aligned}$$

$\nu_2\{c_2c_{n-2}[H^n_{p^t}]\} \geq \alpha(n+1) - 3$

So, $\nu_2\{c_2c_{n-2}[K^n]\} \geq \alpha(n+1) - 3.$

Case 2.4.16 *Let* $(n+1) = 2^{t_1}.p_2^{t_2} \cdots p_k^{t_k}$, *where* $p_2, \ldots, p_k$ *are distinct odd primes;* $t_1, \ldots, t_k \geq 1;\ k \geq 2.$

Then

$$K^n = a_0P^n + a_1H^n_{2^{t_1}} + \sum_{j=2}^{k} a_jH^n_{p_j^{t_j}}.$$

where $a_0, \ldots, a_k \in \mathbb{Z}$

$$\nu_2\{c_2c_{n-2}[P^n]\} = \begin{cases} \geq \alpha(n+1) + 2 & \text{if } t_1 = 1 \\ \alpha(n+1) + 2t_1 - 2 & \text{if } t_1 \geq 2 \end{cases}$$

$$\nu_2\{c_2c_{n-2}[H^n_{2^{t_1}}]\} = \begin{cases} \alpha(n+1) - 2 & \text{if } t_1 = 1 \\ \geq \alpha(n+1) & \text{if } t_1 = 2 \\ \alpha(n+1) - 1 & \text{if } t_1 \geq 3 \end{cases}$$

$\nu_2\{c_2c_{n-2}[H^n_{p_j^{t_j}}]\} \geq \alpha(n+1) + t_j - 1$

$$\text{So, } \nu_2\{c_2c_{n-2}[K^n]\} = \begin{cases} \alpha(n+1) - 2 & \text{if } t_1 = 1 \\ \geq \alpha(n+1) & \text{if } t_1 = 2 \\ \alpha(n+1) - 1 & \text{if } t_1 \geq 3 \end{cases}$$

Case 2.4.17 *Let* $(n+1) = p_1^{t_1} \cdots p_k^{t_k}$, *where* $p_1, \ldots, p_k$ *are distinct, odd primes;* $t_1, \ldots, t_k \geq 1;\ k \geq 2.$

Then

$$K^n = a_0P^n + \sum_{j=1}^{k} a_jH^n_{p_j^{t_j}}.$$

where $a_0, \ldots, a_k \in \mathbb{Z}$

$$\begin{aligned} \nu_2\{c_2c_{n-2}[P^n]\} &= 2\alpha(n) - \alpha(n+2) \\ &= 2\alpha(n+1) - 2 - \alpha(n+2) \end{aligned}$$

$\nu_2\{c_2c_{n-2}[H^n_{p_j^{t_j}}]\} \geq \alpha(n+1) - 3$

So, $\nu_2\{c_2c_{n-2}[K^n]\} \geq \alpha(n+1) - 3 \|$

We can finally summarize the cases above into the following:

Proposition 2.4.18

$$\nu_2\{c_2c_{n-2}[K^n]\} = \begin{cases} \geq \alpha(n+1) - 3 & \textit{if } (n+1) \textit{ is odd} \\ 3 & \textit{if } (n+1) = 4 \\ 1 & \textit{if } (n+1) = 2^{t+1},\ t \geq 2 \\ \alpha(n+1) - 2 & \textit{if } (n+1) = 2.odd \\ \geq \alpha(n+1) & \textit{if } (n+1) = 2^2.odd \\ \alpha(n+1) - 1 & \textit{if } (n+1) = 2^t.odd,\ t \geq 3 \end{cases}$$

We would now like to show that:

$$min\{\nu_2\{c_2c_{n-2}[M^n]\} | M^n \in MU(2n)\} = \rho_2(n) \tag{2.5}$$

As in the previous two cases we will show this in two parts, covering the first part here and dealing with the scond part in general in Section 2.6.

Lemma 2.4.19 $\nu_2\{c_2c_{n-2}[M^n]\} \geq \rho_2(n), \ \forall M^n \in MU(2n)$

Proof

We will use induction.

The result is true for $M^n = K^n$.

Suppose $M^n = U^u \times V^v$ and the result is true for U and V.

Then:

$$\begin{aligned}
\nu_2\{c_2c_{n-2}[U \times V]\} &= \nu_2\{c_u[U]c_2c_{v-2}[V] + c_1c_{u-1}[U]c_1c_{v-1}[V] \\
&\quad + c_2c_{u-2}[U]c_v[V]\}, \text{ by Lemma 1.4.5} \\
&= min^*\{\nu_2\{c_u[U]c_2c_{v-2}[V]\}, \nu_2\{c_1c_{u-1}[U]c_1c_{v-1}[V]\}, \\
&\quad \nu_2\{c_2c_{u-2}[U]c_v[V]\}\}
\end{aligned}$$

Look at:

$$\begin{aligned}
\nu_2\{c_u[U]c_2c_{v-2}[V]\} &= \nu_2\{c_u[U]\} + \nu_2\{c_2c_{v-2}[V]\} \\
&\geq \kappa_1 + \kappa_2
\end{aligned}$$

$$\text{where } \begin{cases} \kappa_1 = min\{k|\alpha(u+k) \leq 2k\}, \text{ by Lemma 2.2.14} \\ \kappa_2 = min\{k|\alpha(v+k) \leq 2k+2\} \end{cases}$$

$$\begin{aligned}
But,\ 2(\kappa_1 + \kappa_2) + 2 &= 2\kappa_1 + (2\kappa_2 + 2) \\
&\geq \alpha(u+\kappa_1) + \alpha(v+\kappa_2) \\
&\geq \alpha(u+v+\kappa_1+\kappa_2) \\
&= \alpha(n+\kappa_1+\kappa_2)
\end{aligned}$$

So, $\kappa_1 + \kappa_2 \geq min\{k|\alpha(n+k) \leq 2k+2\} = \rho_2(n)$.

Similarly,

$\nu_2\{c_1c_{u-1}[U]\} + \nu_2\{c_1c_{v-1}[V]\} \geq \rho_2(n)$ using Lemma 2.3.17.

$\nu_2\{c_2c_{u-2}[U]\} + \nu_2\{c_v[V]\} \geq \rho_2(n)$

This proves the result. ‖

2.5 Calculating $\nu_2\{c_1^2 c_{n-2}[K^n]\}$

We are now going to evaluate $\nu_2\{c_1^2 c_{n-2}[K^n]\}$.. As in previous calculations we shall begin with some lemmas to give us $\nu_2\{c_1^2 c_{n-2}[P^n]\}$ and $\nu_2\{c_1^2 c_{n-2}[H^n_{r,s}]\}$.

Lemma 2.5.1

$$\nu_2\{c_1^2 c_{n-2}[P^n]\} = \begin{cases} \alpha(n) - 1 & \textit{if } n+1 \textit{ is odd} \\ \alpha(n-3) + 2\alpha(n-1) - 2\alpha(n+1) + 6 & \textit{if } n+1 \textit{ is even} \end{cases}$$

Proof

From Lemma 1.4.7, $c_1^2 c_{n-2}[P^n] = (-1)^n (n+1)^2 \binom{2n-2}{n-2}$

Now,

$$\begin{aligned} \nu_2\{(n+1)^2\tbinom{2n-2}{n-2}\} &= 2\{\alpha(n) - \alpha(n+1) + 1\} + \{\alpha(n) + \alpha(n-2) \\ &\quad -\alpha(2n-2)\} \\ &= 3\alpha(n) - 2\alpha(n+1) + \alpha(n-2) - \alpha(n-1) + 2 \end{aligned}$$

$(n+1)$ is odd

$$\begin{aligned} \nu_2\{c_1^2 c_{n-2}[P^n]\} &= 3\alpha(n) - 2\alpha(n) - 2 + \alpha(n-2) - \alpha(n-2) - 1 + 2 \\ &= \alpha(n) - 1 \end{aligned}$$

$(n+1)$ is even

$$\begin{aligned} \nu_2\{c_1^2 c_{n-2}[P^n]\} &= 3\alpha(n-1) + 3 - 2\alpha(n+1) - \alpha(n-1) \\ &\quad +\alpha(n-3) + 1 + 2 \\ &= \alpha(n-3) + 2\alpha(n-1) - 2\alpha(n+1) + 6 \end{aligned}$$

This completes the proof.

Lemma 2.5.2 $c_1^2 c_{n-2}[H^n_{r,s}] = 2\frac{(2r-4)!(2s-4)!}{r!(r-2)!s!(s-2)!}\{-6(r+s)^4 + (r+s)^3(4rs+30) - (r+s)^2(62rs+36) + (r+s)(32r^2s^2+90rs) - 16r^2s^2\}$

<u>**Proof**</u>

Now, $c_1(H_{r,s}) = (-rx - sy)$

$\Rightarrow c_1^2(H_{r,s}) = (r^2x^2 + 2rsxy + s^2y^2)$

Also, $c_{r+s-3}(H_{r,s}) = (-1)^{r+s-1}\Big\{\frac{3}{r}\binom{2r-4}{r-1}\binom{2s-1}{s}x^{r-3}y^s + \frac{r+2s-2}{rs}\binom{2r-3}{r-1}\binom{2s-2}{s-1}x^{r-2}y^{s-1}$
$+ \frac{2r+s-2}{rs}\binom{2r-2}{r-1}\binom{2s-3}{s-1}x^{r-1}y^{s-2} + \frac{3}{s}\binom{2r-1}{r}\binom{2s-4}{s-1}x^ry^{s-3}\Big\}$

(compare proof of Lemma 2.4.2)

So, $c_1^2 c_{r+s-3}(H_{r,s}) =:$

$x^{r-1}y^s\{3r\binom{2r-4}{r-1}\binom{2s-1}{s} + 2(r+2s-2)\binom{2r-3}{r-1}\binom{2s-2}{s-1} + \frac{s}{r}(2r+s-2)\binom{sr-2}{r-1}\binom{2s-3}{s-1}\} +$
$x^ry^{s-1}\{\frac{r}{s}(r+2s-2)\binom{2r-3}{r-1}\binom{2s-2}{s-1} + 2(2r+s-2)\binom{2r-2}{r-1}\binom{2s-3}{s-1} + 3s\binom{2r-1}{r}\binom{2s-4}{s-1}\}$

$= A.x^{r-1}y^s + B.x^ry^{s-1}$, say

Where:

$$
\begin{aligned}
A &= 3r\tfrac{(2r-4)!(2s-1)!}{(r-1)!(r-3)!s!(s-1)!} + 2(r+2s-2)\tfrac{(2r-3)!(2s-2)!}{(r-1)!(r-2)!(s-1)!(s-1)!} \\
&\quad + \tfrac{s}{r}(2r+s-2)\tfrac{(2r-2)!(s2-3)!}{(r-1)!(r-1)!(s-1)!(s-2)!} \\
&= \tfrac{(2r-4)!(2s-3)!}{r!(s-1)!(r-1)!s!}\{3r^2(2s-1)(2s-2)(r-1)(r-2) \\
&\quad +2rs(r+2s-2)(2r-3)(2s-2)(r-1) \\
&\quad +s^2(2r+s-2)(2r-2)(2r-3)(s-1)\} \\
&= \tfrac{2(2r-4)!(2s-3)!}{(r-2)!(s-2)!r!s!}\{10r^3s+12r^2s^2+2rs^3-3r^3-26r^2s \\
&\quad -22rs^2-3s^3+6r^2+12rs+6s^2\} \\
&= \tfrac{2(2r-4)!(2s-3)!}{(r-2)!(s-2)!r!s!}.C, \text{ say.}
\end{aligned}
$$

By symmetry, $B = \frac{2(2r-3)!(2s-4)!}{(r-2)!(s-2)!r!s!}.D$

where $D = \{2r^3s + 12r^2s^2 + 10rs^3 - 3r^3 - 22r^2s - 26rs^2 - 3s^3 + 6r^2 + 12rs + 6s^2\}$

We now add A and B to get:

$\frac{2(2r-4)!(2s-4)!}{(r-2)!(s-2)!r!s!}\{(2s-3).C + (2r-3).D\}$

Now, $(2s-3).C = (20r^3s^2 + 24r^2s^3 + 4rs^4) - (36r^3s + 88r^2s^2 + 50rs^3 + 6s^4) +$

$(9r^3 + 90r^2s + 90rs^2 + 21s^3) - (18r^2 + 36rs + 18s^2)$

While, $(2r-3).D = (4r^4s + 24r^3s^2 + 20r^2s^3) - (6r^4 + 50r^3s + 88r^2s^2 + 36rs^3) +$

$(21r^3 + 90r^2s + 90rs^2 + 9s^3) - (18r^2 + 36rs + 18s^2)$

So, $(2s-3).C + (2r-3).D =$:

$$(4r^4s + 44r^3s^2 + 44r^2s^3 + 4rs^4) - (6r^4 + 86r^3s + 176r^2s^2 + 86rs^3 + 6s^4)$$

$$+(30r^3 + 180r^2s + 180rs^2 + 30s^3) - (36r^2 + 72rs + 36s^2)$$

$$= (r+s)\{(4r^3s + 40r^2s^2 + 4rs^3) - (6r^3 + 80r^2s + 80rs^2 + 6s^3)$$

$$+(30r^2 + 150rs + 30s^2) - (36r + 36s)\} - 16r^2s^2$$

$$= (r+s)^2\{(4r^2s + 4rs^2) - (6r^2 + 74rs + 6s^2) + (30r + 30s) - 36\}$$

$$+(r+s)\{32r^2s^2 + 90rs\} - 16r^2s^2$$

$$= (r+s)^3\{4rs - 6r - 6s + 30\} + (r+s)^2\{62rs - 36\}$$

$$+(r+s)\{32r^2s^2 + 90rs\} - 16r^2s^2.$$

$$= -6(r+s)^4 + (r+s)^3\{4rs + 30\} + (r+s)^2\{62rs + 36\}$$

$$+(r+s)\{32r^2s^2 + 90rs\} - 16r^2s^2$$

This proves the result.||

Notation 2.5.3 *We shall define* $\xi := -6(r+s)^4 + (r+s)^3(4rs+30)$ $- (r+s)^2(62rs+36) + (r+s)(32r^2s^2 + 90rs) - 16r^2s^2$

Lemma 2.5.4 $\nu_2\{c_1^2 c_{n-2}[H^n_{r,s}]\} = \alpha(r) + \alpha(s) - 3 + \nu_2(\xi)$

Proof

$\nu_2\{2\frac{(2r-4)!(2s-4)!}{r!(r-2)!s!(s-2)!}\} =:$

$1 + (2r-4) - \alpha(2r-4) + (2s-4) - \alpha(2s-4) - r + \alpha(r) - (r-2)$

$+ \alpha(r-2) - s + \alpha(s) - (s-2) + \alpha(s-2)$

$= \alpha(r) + \alpha(s) - 3$ ||

We shall now go on to evaluate $\nu_2\{c_1^2c_{n-2}[K_n]\}$. We shall first examine the case where $(n+1)$ is odd, which yields a straightforward answer.

Result 2.5.5 *Let $(n+1)=\eta.\zeta$, where both $\eta \geq 3$ and $\zeta \geq 3$ are odd.*

Let $r=\eta$ and $s=n+1-\eta$.

Then:

$$\begin{aligned}
\nu_2(\xi) &= \nu_2\{-6(r+s)^4+30(r+s)^3+4k\}, \text{ where } k \in Z \\
&\geq 2
\end{aligned}$$

$$\begin{aligned}
So, \nu_2\{c_1^2c_{n-2}[H_{r,s}]\} &\geq \alpha(\eta)+\alpha(n+1-\eta)-3+2 \\
&\geq \alpha(n+1)-1 \\
&= \alpha(n)
\end{aligned}$$

Hence, from Lemma 2.5.4, Result 2.5.5 and Theorem 2.1.6 we have:

Proposition 2.5.6 *If $(n+1)$ is odd then $\nu_2\{c_1^2c_{n-2}[K^n]\}=\alpha(n)-1$.*

We shall continue by examining the most straightforward case when $(n+1)$ is even, namely when $(n+1)$ is a power of 2.

Result 2.5.7 *Let $(n+1)=2^{t+1}$, where $t \geq 1$.*

Then,

$$\begin{aligned}
\nu_2\{c_1^2c_{n-2}[P^n]\} &= 2\alpha(n-1)-2\alpha(n+1)+\alpha(n-3)+6 \\
&= 2t-2+t-1+6 \\
&= 3(t+1) \\
&= 3\alpha(n)
\end{aligned}$$

Result 2.5.8 *Let* $(n+1) = 2^{t+1}$, *where* $t \geq 1$.

Let $r = 2^t = s$

Then:

$$\nu_2\{c_1^2 c_{n-2}[H_{r,s}]\} = \begin{cases} 4 & t = 1 \\ 10 & t = 2 \\ 2\alpha(n) + 1 & t \geq 3 \end{cases}$$

<u>**Proof**</u>

$\nu_2(\xi) =:$

$\nu_2\{-6.2^{4t+4} + 2^{3t+3}(4.2^{2t} + 30) - 2^{2t+2}(62.2^{2t} + 36) + 2^{t+1}(32.2^{4t} + 90.2^{2t}) - 16.2^{4t}\}$

$= \nu_2\{-3.2^{4t+5} + 2^{5t+5} + 15.2^{3t+4} - 31.2^{4t+3} - 9.2^{2t+4} + 2^{5t+6} + 45.2^{3t+2} - 2^{4t+4}\}$

$= \nu_2\{2^{5t+6} + 2^{5t+5} - 3.2^{4t+5} - 2^{4t+4} - 31.2^{4t+3} + 15.2^{3t+4} + 45.2^{3t+2} - 9.2^{2t+4}\}$

$$= \begin{cases} 5 & t = 1 \\ 11 & t = 2 \\ 2t + 4 & t \geq 3 \end{cases}$$

Now,

$$\begin{aligned} \nu_2\{c_1^2 c_{n-2}[H_{r,s}]\} &= \alpha(r) + \alpha(s) - 3 + \nu_2(\xi) \\ &= -1 + \nu_2(\xi) \end{aligned}$$

This gives the result, noting that $\alpha(n) = t + 1$.

Hence from the above results and Theorem 2.1.6 and Lemmas 2.1.7, 2.1.8 and 2.1.9 we have:

Proposition 2.5.9 *If* $(n+1) = 2^{t+1}$

$$\nu_2\{c_1^2 c_{n-2}[K^n]\} = \begin{cases} 4 & if\, t = 1 \\ 9 & if\, t = 2 \\ 2\alpha(n)+1 & if\, t \geq 3 \end{cases}$$

We shall now move on to the final case, when $(n+1)$ is even but not a power of 2. As before we will first examine the particular cases we require individually.

Result 2.5.10 *Let* $(n+1) = 2.\eta$, *where* $\eta \geq 3$ *and odd.*

Then:

$$\begin{aligned} \nu_2\{c_1^2 c_{n-2}[P^n]\} &= \alpha(n-3) + 2\alpha(n-1) - 2\alpha(n+1) + 6 \\ &\geq \alpha(n) - 1 + 2(\alpha(n)-1) - 2\alpha(n) + 6 \\ &= \alpha(n) + 3 \end{aligned}$$

Result 2.5.11 *Let* $(n+1) = 2^t.\eta$, *where* $\eta \geq 3$ *is odd;* $t \geq 2$

Then:

$$\begin{aligned} \nu_2\{c_1^2 c_{n-2}[P^n]\} &= \alpha(n-3) + 2\alpha(n-1) - 2\alpha(n+1) + 6 \\ &= (\alpha(n)-2) + 2(\alpha(n)-1) - 2(\alpha(n)-t+1) + 6 \\ &= \alpha(n) + 2t \end{aligned}$$

Result 2.5.12 *Let* $(n+1) = 2^t.\eta$, *where* $\eta \geq 3$ *is odd;* $t \geq 1$.

Let $r = 2^t$ *and* $s = n + 1 - 2^t$.

Then, $\xi =$:

$$-6.2^{4t}.\eta^4 + 2^{3t}.\eta^3\{2^{2t+2}(\eta-1)+30\} - 2^{2t}\eta^2\{31.2^{2t+1}(\eta-1)+36\}$$

$$+2^t\eta\{2^{4t+5}(\eta-1)^2 + 45.2^{2t+1}(\eta-1)\} - 2^{4t+4}(\eta-1)^2$$

$$= \quad 2^{5t+2}\eta^3(\eta-1) + 2^{5t+5}\eta(\eta-1)^2 - 3.2^{4t+1}\eta^4 - 31.2^{4t+1}\eta^2(\eta-1)$$

$$-2^{4t+4}(\eta-1)^2 + 15.2^{3t+1}\eta^3 + 45.2^{3t+1}\eta(\eta-1) - 9.2^{2t+2}\eta^2$$

$$\Rightarrow \nu_2(\xi) = \begin{cases} \geq 5 & if\ t=1 \\ 2t+2 & if\ t \geq 2 \end{cases}$$

Now,

$$\begin{aligned}
\nu_2\{c_1^2c_{n-2}[H_{r,s}]\} &= \alpha(2^t) + \alpha(2^t.\eta - 2^t) - 3 + \nu_2(\xi) \\
&= \alpha(n+1) - 3 + \nu_2(\xi) \\
&= \alpha(n) - t - 2 + \nu_2(\xi) \\
&= \begin{cases} \geq \alpha(n)+2 & if\ t=1 \\ \alpha(n)+t & if\ t \geq 2 \end{cases}
\end{aligned}$$

Result 2.5.13 *Let $(n+1) = 2^t.\eta.\zeta$, where $\eta \geq 3$ and $\zeta \geq 3$ are both odd; $t \geq 1$.*

Let $r = \eta$ and $s = n+1-\eta$.

Then, $\xi =$:

$$
\begin{aligned}
& -6.2^{4t}.\eta^4.\zeta^4 + 2^{3t}.\eta^3.\zeta^3\{4.\eta^2(2^t.\zeta - 1) + 30\} \\
& -2^{2t}.\eta^2.\zeta^2\{62.\eta^2(2^t\zeta - 1) + 36\} + 2^t.\eta.\zeta\{32.\eta^4(2^{2t}\zeta^2 - 2^{t+1}\zeta + 1) \\
& +90.\eta^2(2^t.\zeta - 1)\} - 16.\eta^4\{2^{2t}\zeta^2 - 2^{t+1}\zeta + 1\} \\
= \; & -3.2^{4t+1}.\eta^4.\zeta^4 + 2^{4t+2}.\eta^5.\zeta^4 - 2^{3t+2}.\eta^5.\zeta^3 \\
& +15.2^{3t+1}.\eta^3.\zeta^3 - 31.2^{3t+1}.\eta^4.\zeta^3 + 2^{3t+5}.\eta^5.\zeta^3 + 31.2^{2t+1}.\eta^4.\zeta^2 \\
& -9.2^{2t+2}.\eta^2.\zeta^2 - 2^{2t+6}.\eta^5.\zeta^2 + 45.2^{2t+1}.\eta^3.\zeta^2 - 2^{2t+4}.\eta^4.\zeta^2 \\
& +2^{t+5}.\eta^5.\zeta - 45.2^{t+1}.\eta^3.\zeta + 2^{t+5}.\eta^4.\zeta - 2^4.\eta^4
\end{aligned}
$$

$$
\Rightarrow \nu_2(\xi) = \begin{cases} 2 & if\; t = 1 \\ 3 & if\; t = 2 \\ \geq 5 & if\; t = 3 \\ 4 & if\; t \geq 4 \end{cases}
$$

Now,

$$
\begin{aligned}
\nu_2\{c_1^2 c_{n-2}[H_{r,s}]\} & = \alpha(\eta) + \alpha(n+1-\eta) - 3 + \nu_2(\xi) \\
& \geq \alpha(n+1) + t - 3 + \nu_2(\xi) \\
& = \alpha(n) - 2 + \nu_2(\xi) \\
& \geq \begin{cases} \alpha(n) & if\; t = 1 \\ \alpha(n)+1 & if\; t = 2 \\ \alpha(n)+3 & if\; t = 3 \\ \alpha(n)+2 & if\; t \geq 4 \end{cases}
\end{aligned}
$$

We can now collect the above results together to give us the following:

Proposition 2.5.14 *Let* $(n+1) = 2^t.\xi$, *where* $\xi \geq 3$ *is odd.*

Then:

$$\nu_2\{c_1^2c_{n-2}[K^n]\} = \begin{cases} \geq \alpha(n) & if\, t = 1 \\ \geq \alpha(n)+1 & if\, t = 2 \\ \alpha(n)+3 & if\, t = 3 \\ \geq \alpha(n)+2 & if\, t \geq 4 \end{cases}$$

We can now finally note, from Propositions 2.5.6, 2.5.9 and 2.5.14 as well as Proposition 2.3.16, the following:

Proposition 2.5.15 $\nu_2\{c_1^2c_{n-2}[K^n]\} \geq \nu_2\{c_1c_{n-1}[K^n]\}$

We would now like to show that:

$$min\{\nu_2\{c_1^2c_{n-2}[M^n]\}|M^n \in MU(2n)\} = \rho_1(n). \qquad (2.6)$$

Note that this result is not covered by the work in [12] and so is original. This gives a clear example of how these methods can be used to tackle individual problems directly.

As in the previous cases we will prove this result in two parts. We will cover the first part here and deal with the second part in Section 2.6

Lemma 2.5.16 $\nu_2\{c_1^2c_{n-2}[M^n]\} \geq \rho_1(n),\ \forall M^n \in MU(2n)$.

<u>Proof</u>

We shall use the, by now familiar, induction.

The result is true for $M^n = K^n$

Suppose $M^n = U^u \times V^v$ and the result is true for U and V.

Now, Lemma 1.4.5 gives us:

$$\begin{aligned}
\nu_2\{c_1^2 c_{n-2}[U \times V]\} &= \nu_2\{c_1^2 c_{u-2}[U]c_v[V] + 2c_1c_{u-1}[U]c_1c_{v-1}[V] \\
&\quad + c_u[U]c_1^2c_{v-2}[V]\} \\
&= min^*\{\nu_2\{c_1^2[U]c_v[V]\}, \nu_2\{2c_1c_{u-1}[U]c_1c_{v-1}[V]\}, \\
&\quad \nu_2\{c_u[U]c_1^2c_{v-2}[V]\}\}
\end{aligned}$$

Look at:

$$\begin{aligned}
\nu_2\{c_1^2c_{u-2}[U]c_v[V]\} &= \nu_2\{c_1^2c_{u-2}[U]\} + \nu_2\{c_v[V]\} \\
&\geq \kappa_1 + \kappa_2
\end{aligned}$$

where $\begin{cases} \kappa_1 = min\{k|\alpha(u+k) \leq 2k+1\} \\ \kappa_2 = min\{k|\alpha(v+k) \leq 2k\}, \text{ by Lemma 2.2.14} \end{cases}$

$$\begin{aligned}
But,\ 2(\kappa_1+\kappa_2)+1 &= (2\kappa_1+1)+2\kappa_2 \\
&\geq \alpha(u+\kappa_1)+\alpha(v+\kappa_2) \\
&\geq \alpha(u+v+\kappa_1+\kappa_2) \\
&= \alpha(n+\kappa_1+\kappa_2)
\end{aligned}$$

Similarly we can show:

$\nu_2\{2c_1c_{u-1}[U]c_1c_{v-1}[V]\} \geq \rho_1(n)$, using Lemma 2.3.17

$\nu_2\{c_u[U]c_1^2c_{v-2}[V]\} \geq \rho_1(n)$.

This proves the result.||

2.6 A Construction of Required Manifolds

In this section we will construct the particular manifolds, M^n, which realize the divisibility property that $\nu_2\{c_t c_{n-t}[M^n]\} = \rho_t(n)$. This will serve to complete the proofs of Sections 2.2, 2.3 and 2.4 that $min\{\nu_2\{c_t c_{n-t}[M^n]\} | M^n \in MU(2n)\} = \rho_t(n)$, in the cases where $t = 0, 1, 2$.

We will then go on to show that, in the case $t = 1$, these manifolds also have the property that $\nu_2\{c_1^2 c_{n-2}[M^n]\} = \rho_1(n)$. This will complete the proof of Section 2.5 that $min\{\nu_2\{c_1^2 c_{n-2}[M]| M^n \in MU(2n)\} = \rho_1(n)$.

We will use the following special cases of Propositions 2.2.13 and 2.3.16:

Lemma 2.6.1 *Let $\alpha_i = 2^r + 2^s - 1$, where $r \neq s$ and $r, s \geq 1$.*

Let $\beta_j = 2^t$, where $t \geq 1$.

Then:

$\nu_2\{c_{\alpha_i}[K^{\alpha_i}]\} = 1$

$\nu_2\{c_{\beta_j}[K^{\beta_j}]\} = 1$

$\nu_2\{c_1 c_{\alpha_i - 1}[K^{\alpha_i}]\} \geq 2$

$\nu_2\{c_1 c_{\beta_j - 1}[K^{\beta_j}]\} = 0$

$\nu_2\{c_u[K^u]\} \geq 1, \ \forall u \in \mathbb{N}$.

<u>**Construction**</u>

Let $\rho_t(n) = min\{k | \alpha(n + k) \leq 2k + t\} := \kappa$.

<u>$n+\kappa$ **is even**</u>

Now, from the definition of κ we can write:

$$n+\kappa = 2^{a_1}+\cdots+2^{a_{2\kappa}}+2^{b_1}+\cdots+2^{b_t} \tag{2.7}$$

where $a_1,\ldots,a_{2\kappa},b_1,\ldots,b_t \geq 1$ and $a_i \neq a_{2\kappa+1-i}$, $\forall 1\leq i\leq \kappa$.

Put $\alpha_i := 2^{a_i}+2^{a_{2\kappa+1-i}}-1$, $\beta_j := 2^{b_j}$ for $1\leq i\leq\kappa$, $1\leq j\leq t$.

Put:

$$M^n := K^{\alpha_1}\times\cdots\times K^{\alpha_\kappa}\times K^{\beta_1}\times\cdots\times K^{\beta_t} \tag{2.8}$$

<u>**Proof that** $\nu_2\{c_t c_{n-t}[M^n]\}=\kappa$</u>

Now, Lemma 1.4.5 gives us:

$$c_t c_{n-t}[M^n] = \sum_{r,s} c_{i_r} c_{\alpha_r - i_r}[K^{\alpha_r}] c_{j_s} c_{\beta_s - j_s}[K^{\beta_s}] \tag{2.9}$$

where the summation is over every $i_1+\cdots+i_\kappa+j_1+\cdots+j_t=t$.

We will show that $2^{\kappa+1}$ divides all but one of these terms, with that one being an odd multiple of 2^κ. This will prove the result.

Now, $2|c_u[K^u]$, $\forall u$ so if $\kappa+1$ of the i's and j's are zero, $2^{\kappa+1}$ will divide that term. The only way of avoiding this is to have t of the i's and j's equal to 1 and the rest equal to 0.

But, $2|c_1 c_{\alpha_i-1}[K^{\alpha_i}]$, so if any of the i's are equal to 1 then 2 will divide that particular element and 2 will divide each of the κ elements where the i's and j's are equal to zero. So $2^{\kappa+1}$ will divide the whole term.

Hence $2^{\kappa+1}$ divides every term of our sum except one, when all the i's are zero and all the j's are one. Then $\nu_2\{c_{i_r}[K^{\alpha_r}]\} = 1$ and $\nu_2\{c_1 c_{j_s-1}[K^{\beta_s}]\} = 0$. So this term is an odd multiple of 2^κ.

This completes the proof||.

<u>$n + \kappa$ **is odd**</u>

As before, from the definition of κ we can write:

$$n + \kappa = 2^{a_1} + \cdots + 2^{a_{2\kappa}} + 2^{b_1} + \cdots + 2^{b_t - 1} + 1 \tag{2.10}$$

where $a_i, b_j \geq 1,\ \forall 1 \leq i \leq \kappa,\ 1 \leq j \leq t$ and $a_i \neq a_{2\kappa+1-i},\ \forall 1 \leq i \leq \kappa$.

Put $\alpha_i := 2^{a_i} + 2^{a_{2\kappa+1-i}} - 1,\ 1 \leq i \leq \kappa - 1; \alpha_\kappa := 2^{a_\kappa} + 2^{a_{\kappa+1}}; \beta_j := 2^{b_j},\ 1 \leq j \leq t-1$.

Then:

$$M^n := K^{\alpha_1} \times \cdots \times K^{\alpha_\kappa} \times K^{\beta_1} \times \cdots \times K^{\beta_{t-1}} \tag{2.11}$$

<u>**Proof that** $\nu_2\{c_t c_{n-t}[M^n]\} = \rho_t(n)$</u>

Similar to the previous proof, this time the *min** term is:

$c_{\alpha_1}[K^{\alpha_1}] \cdots c_{\alpha_{\kappa-1}}[K^{\alpha_{\kappa-1}}] c_1 c_{\alpha_\kappa - 1}[K^{\alpha_\kappa}] c_1 c_{\beta_1 - 1}[K^{\beta_1}] \cdots c_1 c_{\beta_{t-1}-1}[K^{\beta_{t-1}}]$.||

This completes the construction.

We will now go on to show that in fact these same manifolds, in the case that

$t = 1$ have the property that $\nu_2\{c_1^2 c_{n-2}[M^n] = \rho_1(n)$.

$n + \kappa$ is even

We shall use the same construction of M^n as in equation 2.8.

The proof that this is the required manifold is also similar to the previous proof.

In this case the *min** term is:

$c_{\alpha_1}[K^{\alpha_1}] \cdots c_{\alpha_\kappa}[K^{\alpha_\kappa}] c_1^2 c_{\beta_1 - 2}[K^{\beta_1}]||$

$n + \kappa$ is odd

We shall use the same construction of M^n as in equation 2.11.

The proof that this is the required manifold is also similiar to the previous proof.

In this case the *min** term is:

$c_{\alpha_1}[K^{\alpha_1}] \cdots c_{\alpha_{\kappa-1}}[K^{\alpha_{\kappa-1}}] c_1^2 c_{\alpha_\kappa - 2}[K^{\alpha_\kappa}]$,

where $\nu_2\{c_1^2 c_{\alpha_\kappa - 2}[K^{\alpha_\kappa}]\} = 1$, from Lemma 2.5.6.

This completes the construction.

This construction, along with Section 2.5, proves the following lemma:

Lemma 2.6.2 $\nu_2\{hcf\{c_1^2 c_{n-2}[M^n] | M^n \in MU(2n)\}\} = \rho_1(n)$

$t = 1$ have the property that $\nu_2\{c_1^2 c_{n-2}[M^n] = \rho_1(n)$.

$n + \kappa$ is even

We shall use the same construction of M^n as in equation 2.8.

The proof that this is the required manifold is also similar to the previous proof.

In this case the min^* term is:

$c_{\alpha_1}[K^{\alpha_1}] \cdots c_{\alpha_\kappa}[K^{\alpha_\kappa}] c_1^2 c_{\beta_1 - 2}[K^{\beta_1}]||$

$n + \kappa$ is odd

We shall use the same construction of M^n as in equation 2.11.

The proof that this is the required manifold is also similiar to the previous proof.

In this case the min^* term is:

$c_{\alpha_1}[K^{\alpha_1}] \cdots c_{\alpha_{\kappa-1}}[K^{\alpha_{\kappa-1}}] c_1^2 c_{\alpha_\kappa - 2}[K^{\alpha_\kappa}]$,

where $\nu_2\{c_1^2 c_{\alpha_\kappa - 2}[K^{\alpha_\kappa}]\} = 1$, from Lemma 2.5.6.

This completes the construction.

This construction, along with Section 2.5, proves the following lemma:

Lemma 2.6.2 $\nu_2\{hcf\{c_1^2 c_{n-2}[M^n] | M^n \in MU(2n)\}\} = \rho_1(n)$

2.7 Calculating $hcf\{c_1c_{n-1}+\frac{n}{2}(3n-5)c_n\}[M^n] | M \in MU(2n)\}$

This final section of this chapter serves mainly as an example of how the direct approach used previously in this chapter can be used to give relatively straightforward solution to particular problems. Here a particular result of Libgober and Wood [6] is verified and an improvement to the result is made.

The working in this section will all involve tangent chern classes. Keeping to convention no change in notation will be made.

Proposition 2.7.1 (Libgober and Wood) $12 | \{c_1c_{n-1}+\frac{n}{2}(3n-5)c_n\}[M^n]$, $\forall M^n \in MU(2n)$

Proof

The proof will be in three parts:

i) $12 | \{c_1c_{n-1}+\frac{n}{2}(3n-5)c_n\}[P^n]\ \forall n$

[10] gives us: $c(P^n) = (1+x)^{n+1}/x^{n+1} \equiv 0$.

So,

$$\begin{aligned}
\{c_1c_{n-1}+\tfrac{n}{2}(3n-5)c_n\}[P^n] &= \binom{n+1}{1}\binom{n+1}{n-1}+\tfrac{n}{2}(3n-5)\binom{n+1}{n} \\
&= \tfrac{n}{2}(n+1)^2+\tfrac{n}{2}(n+1)(3n-5) \\
&= \tfrac{n}{2}(n+1)(n+1+3n-5) \\
&= 2(n-1)n(n+1)
\end{aligned}$$

And $12|2(n-1)n(n+1)\forall n||$

ii) $12|\{c_1c_{n-1} + \frac{n}{2}(3n-5)c_n\}[H^n_{r,s}]\ \forall H.$

As above, [10] gives us:

$c(H_{r,s}) = (1+x)^{r+1}(1+y)^{s+1}(1+x+y)^{-1}/x^{r+1} \equiv y^{s+1} \equiv x^r y^s \equiv 0.$

WLOG let $r \geq s$, then we have:

$$\begin{aligned} c(H_{r,s}) &= (1+y)^{s+1}\tfrac{(x+1)^{r+1}}{x+(y+1)} \\ &= (1+y)^{s+1}\{x^r + rx^{r-1} - yx^{r-1} + \tfrac{1}{2}r(r-1)x^{r-2} + (1-r)yx^{r-2} \\ &\quad + y^2x^{r-2} + o(x^{r-3},\dots,x) + \tfrac{R}{x+(y+1)}\} \end{aligned}$$

where $R = (x+1)^{r+1}|_{x=-y-1} = (-y)^{r+1} \equiv 0.$

Hence, $c_{r+s-1}(H_{r,s}) = x^r y^{s-1}\binom{s+1}{s-1} + x^{r-1}y^s\{r\binom{s+1}{s} - \binom{s+1}{s-1}\}$

$$\begin{aligned} \Rightarrow c_{r+s-1}[H_{r,s}] &= \tfrac{1}{2}s(s+1) + r(s+1) - \tfrac{1}{2}s(s+1) \\ &= r(s+1) \end{aligned}$$

Also,

$$\begin{aligned} c_{r+s-2}(H_{r,s}) &= x^r y^{s-2}\binom{s+1}{s-2} + x^{r-1}y^{s-1}\{r\binom{s+1}{s-1} - \binom{s+1}{s-2}\} \\ &\quad + x^{r-2}y^s\{\tfrac{1}{2}r(r-1)\binom{s+1}{s} + (1-r)\binom{s+1}{s-1} + \binom{s=1}{s-2} \\ &= x^r y^{s-2}\tfrac{1}{6}(s-1)s(s+1) + x^{r-1}y^{s-1}\{\tfrac{1}{2}rs(s+1) \\ &\quad - \tfrac{1}{6}(s-1)s(s+1)\} + x^{r-2}y^s\{\tfrac{1}{2}r(r-1)(s+1) \\ &\quad + \tfrac{1}{2}(1-r)s(s+1) + \tfrac{1}{6}(s-1)s(s+1)\} \end{aligned}$$

And, $c_1(H_{r,s}) = rx + sy.$

$$\begin{aligned}
\Rightarrow c_1 c_{r+s-2}[H_{r,s}] &= \tfrac{1}{6}(s-1)s^2(s+1) + \tfrac{1}{2}r^2 s(s+1) - \tfrac{1}{6}rs(s-1)(s+1) \\
&\quad + \tfrac{1}{2}rs^2 s + 1 - \tfrac{1}{6}(s-1)s^2(s+1) + \tfrac{1}{2}r^2(r-1)(s+1) \\
&\quad + \tfrac{1}{2}(1-r)rs(s+1) + \tfrac{1}{6}r(s-1)s(s+1) \\
&= \tfrac{1}{2}r(s+1)\{rs + s^2 + r(r-1) + s(1-r)\} \\
&= \tfrac{1}{2}r(s+1)(r^2 + s^2 - r + s)
\end{aligned}$$

So we can put all this together to get:

$$\begin{aligned}
&\{c_1 c_{r+s-2} + \tfrac{1}{2}(3r+3s-8)(r+s-1)c_{r+s-1}\}[H_{r,s}] =: \\
&\quad \tfrac{1}{2}r(s+1)(r^2+s^2-r+s) + \tfrac{1}{2}(r+s-1)(3r+3s-8)r(s+1) \\
&= \tfrac{1}{2}r(s+1)(r^2+s^2-r+s+3r^2+3s^2+6rs-11r-11s+8) \\
&= r(s+1)(2r^2+2s^2-6r+5s+3rs+4)
\end{aligned}$$

Now we just have to show that this is not divisible by 12 for all $r, s \geq 2$.

Let $A = r$, $B = (s+1)$, $C = (2r^2 + 2s^2 - 6r - 5s + 3rs + 4)$.

<u>mod2</u>

$(r,s) = (0,1) \Rightarrow A \equiv 0,\ B \equiv 0$

$(r,s) = (0,0) \Rightarrow A \equiv 0,\ C \equiv 0$

$(r,s) = (1,1) \Rightarrow B \equiv 0,\ C \equiv 0$

$(r,s) = (1,0) \Rightarrow C \equiv 0 \bmod 4$

<u>mod3</u>

$r = 0 \Rightarrow A \equiv 0$

$s = 2 \Rightarrow B \equiv 0$

otherwise, $C \equiv 0.\|$

iii) It will now suffice to prove the following:

Lemma 2.7.2 $12|\{c_1c_{m-1} + \frac{m}{2}(3m-5)c_m\}[M^m]$ *and* $12|\{c_1c_{n-1}$

$+ \frac{n}{2}(3n-5)c_n\}[N^n] \Rightarrow 12|\{c_1c_{m+n-1} + \frac{m+n}{2}(3m+3n-5)c_{m+n}\}[M^m \times N^n]$

Proof

$$\begin{aligned}
\{c_1c_{m+n-1} + \tfrac{m+n}{2}(3m+3n-5)c_{m+n}\}[M^m \times N^n] =: \\
& c_1c_{m-1}[M]c_n[N] + c_m[M]c_1c_{n-1}[N] + \{\tfrac{m}{2}(3m+3n-5) \\
& +\tfrac{n}{2}(3m+3n-5)\}c_m[M]c_n[N] \\
= \;& c_1c_{m-1}[M]c_n[N] + \tfrac{m}{2}(3m-5)c_m[M]c_n[N] + \tfrac{m}{2}3nc_m[M]c_n[N] \\
& +c_m[M]c_1c_{n-1}[N] + \tfrac{n}{2}(3n-5)c_m[M]c_n[N] + \tfrac{n}{2}3mc_m[M]c_n[N] \\
= \;& c_n[N]\{c_1c_{m-1}[M] + \tfrac{m}{2}(3m-5)c_m[M]\} + c_m[M]\{c_1c_{n-1} \\
& +\tfrac{n}{2}(3n-5)c_n[N]\} + 3mnc_m[M]c_n[N] \\
\equiv \;& 0 \bmod 12
\end{aligned}$$

by the hypotheses and the fact that $c_k[K^k]$ is even $\Leftarrow k$ is odd.||

This completes the proof of Proposition 2.7.1

We can however say more than this, in the next five lemmas we shall calculate $hcf\{c_1c_{n-1} + \frac{n}{2}(3n-5)c_n\}[M^n]$ explicitly.

Lemma 2.7.3 $\forall n \; \exists \; M^n$ *s.t.* $\{c_1c_{n-1} + \frac{n}{2}(3n-5)c_n\}[M^n] = 3k$, *where* $3 \not| k$.

Proof

For ease of notation we will define the following:

$$\begin{aligned} X(r,s) &:= \{c_1c_{r+s-1} + \tfrac{1}{2}(r+s)(3r+3s-5)c_{r+s}\}[P^r \times P^s] \\ &= (s+1)2(r-1)r(r+1) + (r+1)2(s-1)s(s+1) \\ &\quad +3rs(r+1)(s+1) \\ &= (r+1)(s+1)(2r^2+2s^2-2r-2s+3rs) \end{aligned}$$

$\underline{n \equiv 0 \bmod 3}$

Put $r = 3k+1, \ s = 3l+2$, where $k, l \in \mathbb{Z}, \ 3 \not| l+1$.

Then:

$$\begin{aligned} X(r,s) &= (3k+2)(3l+3)\{18k^2+12k+2-6k-2+18l^2+24l+8 \\ &\quad -6l-4+3(3k+1)(3l+2)\} \\ &= 3(3k+2)(l+1)\{3[6k^2+2k+6l^2+6l+(3k+1)(3l+2)] \\ &\quad +4\} \end{aligned}$$

Similarly if $n \equiv 1 \bmod 3$, put $r = 3k, \ s = 3l+1$, where $3 \not| l-k$.

If $n \equiv 2 \bmod 3$, put $r = 3k, \ s = 3l+2$, where $3 \not| l+1$.||

Lemma 2.7.4 *For any prime $p \geq 5$ $\forall n$ $\exists M^n$ s.t.*

$p \not| \{c_1c_{n-1} + \frac{n}{2}(3n-5)c_n\}[M^n]$

<u>Proof</u>

Suppose $n = \lambda p + \epsilon, \ 0 \leq \epsilon < p$

Put $r = \lambda p, \ s = \epsilon$.

Then $X(r,s) = (\lambda p+1)(\epsilon+1)\{p(2\lambda^2 p - 2\lambda + 3\lambda\epsilon) + 2\epsilon(\epsilon-1)\}$

So $p \not| X(r,s)$ unless $\epsilon = 0, 1$, or $p-1$.

If $\epsilon = 0$ put $r = \lambda p - 2, \ s = 2$.

If $\epsilon = 1$ put $r = \lambda p - 3,\ s = 2$.

If $\epsilon = p - 1$ put $r = \lambda p + 1,\ s = p - 2.\|$

Lemma 2.7.5 *$\forall n\ \exists M^n$ s.t.$\{c_1c_{n-1} + \frac{n}{2}(3n - 5)c_n\}[M^n] = 4k$, where k is odd, unless $n \equiv 1 \bmod 4$.*

Proof

If $n \equiv 0 \bmod 4$ put $r \equiv 2 \bmod 4,\ s \equiv 2 \bmod 4$.

Then $X(r, s) = 4.odd$.

If $n \equiv 2 \bmod 4$ put $r \equiv 0 \bmod 4,\ s \equiv 2 \bmod 4$.

If $n \equiv 3 \bmod 4$ put $r \equiv 2 \bmod 4,\ s \equiv 1 \bmod 4.\|$

Lemma 2.7.6 *$\forall n \equiv 1 \bmod 4$, $\exists M^n$ s.t. $\{c_1c_{n-1} + \frac{n}{2}(3n-5)c_n\}[M^n] = 8k$, where k is odd.*

Proof

For $n \equiv 1 \bmod 8$ put $r \equiv 4 \bmod 8,\ s \equiv 5 \bmod 8$.

For $n \equiv 5 \bmod 8$ put $r \equiv 2 \bmod 8,\ s \equiv 3 \bmod 8.\|$

We have thus shown that $hcf\{(c_1c_{n-1} + \frac{n}{2}(3n - 5)c_n)[M^n] | M^n \in MU(2n)\}|24$, when $n \equiv 1 \bmod 4$. We would now like to show that 24 divides this expression for every such manifold. To do this we will adopt a slightly different track and go back to Libgober and Wood's original paper, [6].

Lemma 2.7.7 *$24|\{c_1c_{n-1} + \frac{n}{2}(3n - 5)c_n\}[M^n]$, $\forall M^n \in MU(2n)$, where $n \equiv 1 \bmod 4$*

Proof

We shall u·· the notation given in [4], §15.5(10).

Here W is a complex analytic vector bundle over the compact complex manifold V^n. We let $H^*(V,W)$ be the cohomology groups of V with coefficients in the sheaf of germs of local holomorphic sections of W. We can define the Euler-Poincaré characteristic by:

$$\chi(V,W) = \sum_{i=0}^{n}(-1)^i dim H^i(V,W).$$

Letting $\lambda^p T$ denote the p'th exterior product of the cotangent bundle of V, define:

$$\begin{aligned} \chi^p(V,W) &= \chi(V, W\otimes\lambda^p T) \\ &= \textstyle\sum_{q=0}^{n}(-1)^q \mathrm{dim} H^q(V, W\otimes\lambda^p T) \\ &= \textstyle\sum_{q=0}^{n}(-1)^q h^{p,q}(V,W) \end{aligned}$$

where $h^{p,q}(V,W) :=\mathrm{dim}H^q(V, W\otimes\lambda^p T)$.

In particular, if $W=1$, we have:

$$\chi^p(V,1) := \chi^p(V) = \sum_{q=0}^{n}(-1)^q h^{p,q}(V)$$

Then ([6], §2.3) gives us:

$$\sum_{p=2}^{n}(-1)^p\binom{p}{2}\chi^p[M^n] = \frac{1}{12}\{c_1c_{n-1}+\frac{n}{2}(3n-5)c_n\}[M^n]$$

Now,

$$
\begin{aligned}
\textstyle\sum_{p=2}^{n}(-1)^p\binom{p}{2}\chi^p[M^n] &\equiv \textstyle\sum_{p\in I}(-1)^p\chi^p[M^n] \bmod 2 \\
&\quad \text{where } I=\{2,3,6,7,\ldots,n-3,n-2\} \\
&= \textstyle\sum_{p\in I}(-1)^p\sum_{q=0}^{n}(-1)^q h^{p,q}[M^n] \\
&= \textstyle\sum_{p\in I}\sum_{q=0}^{n}(-1)^{p+q}h^{p,q}[M^n]
\end{aligned}
$$

Now, $h^{p,q}[M^n]=h^{n-p,n-q}[M^n]\Rightarrow \chi^i=\chi^{n-i},\ \forall 2\le i\le[\frac{n}{2}]$

Hence,

$$\sum_{p\in I}\sum_{q=0}^{n}(-1)^{p+q}h^{p,q}[M^n]\equiv 0 \bmod 2$$

This proves the result $\|$.

We can now summarize the above five lemmas in the following:

Proposition 2.7.8

$$hcf\{\{c_1c_{n-1}+\frac{n}{2}(3n-5)c_n\}[M^n] \mid M^n\in MU(2n)\} = \begin{cases} 12 & \textit{if } n\not\equiv 1 \bmod 4 \\ 24 & \textit{if } n\equiv 1 \bmod 4 \end{cases}$$

Chapter 3

3.1 Preliminaries

In this chapter we will be concerned with odd primes that divide $hcf\{c_I[M^n]|M^n \in MU(2n)\}$. As before the case where $l(I) = 2$ has already been looked at, in [2], and our main theorem will be a generalization of the ideas contained there.

From looking at some simple examples it may be tempting to guess that no odd prime is ever involved in an h.c.f., but the following refutes that immediately:

$$hcf\{c_2c_1^2[M^4]|M \in MU(8)\} = 3 \tag{3.1}$$

This is the only case among simple chern numbers where an odd prime is involved in the h.c.f., but it shows that we cannot hope to eliminate odd primes from our work completely.

In [2] the following theorem is presented:

Theorem 3.1.1 (Barton and Rees) *No odd prime is involved in* $hcf\{c_r c_{n-r}[M^n] | M^n \in MU(2n)\}$.

The proof of this result involves looking at the matrix $\left[c_I[P^J]\right]$, where I is a partition of n of length ≤ 2 and P^J is some cross-product of projective spaces. However, it is our contention that there is a gap in the proof as it is presented and we complete that gap here.

In [2], example(6) the following matrix is examined: $\left[c_I[P^J]\right]$, where I is a partition of the form $r(n-r)$ and J is a partition of the form $1^{n-3i}.3^i$. We shall note that this is not a square matrix: there are $\left[\frac{n+2}{2}\right]$ such I and $\left[\frac{n+3}{3}\right]$ such J. We can make this into a square matrix by only considering partitions I where $0 \leq r \leq [\frac{n}{3}]$. This example, as it stands, then proves:

$$3 \not| hcf\{c_r c_{n-r}[M^n] | M \in MU(2n)\}, 0 \leq r \leq \left[\frac{n}{3}\right] \tag{3.2}$$

We still require to show:

$$3 \not| hcf\{c_r c_{n-r}[M^n] | M \in MU(2n)\}, \left[\frac{n}{3}\right] < r \leq \left[\frac{n}{2}\right] \tag{3.3}$$

However, the above example provides manifolds N^i such that:

$$\sum_{j\geq 0} c_j c_{n-j}[N^i]t^j = (1+t)^{n-2i}t^i, \ for\ 0 \leq i \leq \left[\frac{n}{3}\right] \tag{3.4}$$

In particular, $c_r c_{n-r}[N^i] = \binom{n-2i}{r-i}$.

We can now formulate our problem in number theoretic terms:

Given n, given r s.t. $[\frac{n}{3}] < r \leq [\frac{n}{2}]$ we want to find i s.t.:

i) $0 \leq i \leq [\frac{n}{3}]$,

ii) $3 \not| \binom{n-2i}{r-i}$.

<u>n even</u>

Let t be s.t. $3^t - 1 \leq n < 3^{t+1} - 1$.

Put $2i = n - 3^t + 1$.

Then:

$$\begin{aligned} (n-2i)\cdots(n-i-r+1) &= (3^t-1)\cdots(3^t-r+i) \\ &\equiv (-1)^{r-i}1.2.3.\ldots.(r-i) \bmod 3^t \end{aligned}$$

$\Rightarrow \binom{n-2i}{r-i} \equiv (-1)^{r-i} \bmod 3^t$

<u>*Check* :</u> $n < 3^{t+1} - 1 \Rightarrow \frac{n}{3} < 3^t - \frac{1}{3} \Rightarrow 3^t > \frac{n}{3} + \frac{1}{3}$

$2i = n - 3^t + 1 < n - \frac{n}{3} - \frac{1}{3} + 1 = 2(\frac{n}{3} + \frac{1}{3})$

$\Rightarrow i < \frac{n}{3} + \frac{1}{3} \Rightarrow i \leq [\frac{n}{3}]$.

<u>n odd</u>

Let t be s.t. $2.3^t - 1 \leq n < 2.3^{t+1} - 1$.

Put $2i = n - 2.3^t + 1$.

The argument is then similar to the above.||

This now completes the proof of Theorem 3.1.1

3.2 Symmetric Functions

Our aim for this chapter will be to generalize the work in [2] to produce a result which will be valid for all chern numbers. In order to do this we will require some knowledge of symmetric functions, which we will now consider. A much deeper study of this work is given in, for example, [8], where proofs of the following will also be found.

A symmetric function is a polynomial $f(x_1, x_2, x_3, \ldots)$ that is constant under permutation of variables. We shall use the notation

$$\sum x_1^{i_1} \cdots x_r^{i_r}$$

to indicate the smallest symmetric function which includes the term $x_1^{i_1} \cdots x_r^{i_r}$. The set of all symmetric functions forms a ring which is graded according to the (homogeneous) degree. Each level has dimension $p(n)$, the number of partitions of n. There are many different bases we can consider, in particular we will look at those consisting of the following:

$$m_I := \sum x_1^{i_1} \cdots x_r^{i_r}$$

$$e_k := \sum x_1 \cdots x_k \ (= m_{(1^k)}); \ e_I := e_{i_1} \cdots e_{i_r}$$

where I is the partition $i_1 + \cdots + i_r = n$.

Here both $\{m_I | I \vdash n\}$ and $\{e_I | I \vdash n\}$ are bases for the n–level of the ring of symmetric functions. The functions m_I are known as the monomial symmetric functions while the functions e_I are the elementary symmetric functions.

We can change basis, $\{m_I\} \to \{e_I\}$. The matrix which does this, call it $A(m_I, e_I)$, is triangular and has diagonal elements ± 1.

3.3 Odd Primes in $c_I[M^n]$

We can now use the symmetric functions described in Section 3.2 to help us examine chern numbers. The basic, motivating result tells us that the monomial symmetric functions break down over added partitions in exactly the same way that chern numbers break down over cross-products, namely that:

$$D_r(M) = \sum_{I|l(I)\leq r} c_I[M^n]m_I$$

is a ring homomorphism : $MU(*) \rightarrow \mathbb{Z}[symmetric\ functions]$, for any $r \leq n$. Restricting this sum by the length of the partitions involved is important to what follows. To make this a natural thing to do we will order the terms by their indexing partitions as follows:

given two, non-equal partitions $I = (i_1 + \cdots + i_r = n)$ and $J = (j_i + \cdots j_s = n)$ we will put $I < J$ if $l(I) < l(J)$. If $l(I) = l(J)$ we will use the normal, reverse lexicographic ordering, namely $I < J$ if $i_k > j_k$, where $i_t = j_t\ \forall\ 1 \leq t \leq (k-1)$. With the above ordering, restricting the sum $D_n(M)$ to $D_r(M)$ simply involves deleting some terms from the end.

We shall prove:

Lemma 3.3.1

$$\sum_{I\vdash(m+n),l(I)\leq r} c_I[M^m \times N^n]m_I = \left(\sum_{J\vdash m,l(J)\leq r} c_J[M]m_J\right)\left(\sum_{K\vdash n,l(K)\leq r} c_K[N]m_K\right)$$

Proof

We shall first look at the following:

$$\left(\sum_{J\vdash m,l(J)\leq r} c_J[M]x_1^{j_1}\cdots x_{m+n}^{j_{m+n}}\right)\left(\sum_{K\vdash n,l(K)\leq r} c_K[N]x_1^{k_1}\cdots x_{m+n}^{k_{m+n}}\right), \tag{3.5}$$

summed over ordered partitions J and K.

Here, given an ordered partition $I = (i_1, \ldots, i_{m+n})$ of weight $(m+n)$, the coefficient of $x_1^{i_1}\cdots x_{m+n}^{i_{m+n}}$ is:

$$\sum_{J+K=I,\ l(J),\ l(K)\leq r} c_J[M]c_K[N]$$

summed over ordered partitions J and K.

Hence, by Lemma 1.4.3 we have:

$$(3.5) = \sum_{I\vdash(m+n),l(I)\leq r} c_I[M\times N]x_1^{i_1}\cdots x_{m+n}^{i_{m+n}} \tag{3.6}$$

where the summation is over all ordered partitions.

We can now drop the ordering of the partitions in (3.6), and we find that the coefficient of $c_J[M]$, say, is $\sum_{\pi(j_1,\ldots,j_{m+n})} x_1^{j_1}\cdots x_{m+n}^{j_{m+n}}$, summed over all permutations π, $= m_J$.

The result follows.||

We now come to the main result of the chapter.

Theorem 3.3.2 *$hcf\{c_I[M^n] | M \in MU(2n)\}$ is not divisible by any prime $p > r+1$, where $l(I) = r$.*

Proof

We shall examine the matrix $\Delta = \left[c_I[P^J]\right]$,

where: I varies over all partitions of length $\leq r$

J varies over all partitions of height $\leq r$

Note that $\#\{I\} = \#\{J\}$ as one set is the dual of the other. So Δ is a square matrix.

Now, we know that if $p|$any column of Δ then $p|det\Delta$. So by the contrapositive, if $p \not| det\Delta$ then $p \not|$any column of Δ and so $p \not| hcf\{c_I[M^n] | M \in MU(2n)\}$ for any I in the above set.

We shall show that $det\Delta$ is a product only of primes $\leq (r+1)$, and this will prove the result.

Now, the rows of Δ are the co-efficients of $D_r(P_J)$.

If we multiply Δ by the matrix $A(m_I, e_I)$ from Section 3.2 we get Δ', say.

$Det\{A(m_I, e_I)\} = \pm 1 \Rightarrow det\Delta = \pm det\Delta'$.

The rows of Δ' are the co-efficients of $D_r(P^J)$ in terms of the $\{e_I\}$.

We shall show first of all that Δ' is diagonal.

Let J be the partition $(j_1 + \cdots + j_s = n)$, let $I_k \vdash j_k$, then:

$$\begin{aligned}
\sum_{l(I)\leq r} c_I[P^J]m_I &= \prod_{k=1}^{s}\left\{\sum_{l(I_k)\leq r} c_{I_k}[P^{j_k}]m_{I_k}\right\} \text{ by Lemma (3.3.1)}\\
&= \prod_{k=1}^{s}\left\{\sum_{I_k\vdash j_k} \kappa'(I_k)m_{I_k}\right\}, \text{ where } \kappa'(I_k) \in Z\\
&= \prod_{k=1}^{s}\left\{\sum_{I_k\vdash j_k} \kappa(I_k)e_{I_k}\right\}, \text{ where } \kappa(I_k) \in Z\\
&= \sum_{I_k\vdash j_k} \kappa(I_1)\cdots\kappa(I_s)e_{I_1\wedge\cdots\wedge I_s}
\end{aligned}$$

In particular, if $I > J$ then e_I does not appear in this expression.

Furthermore, the diagonal elements of Δ' are equal to $k_J = \kappa\left((j_1)\right)\cdots\kappa\left((j_s)\right)$.

We shall now show that $\kappa\left((j_l)\right) = -(j_l+1)$. Since every $j_l \leq r$, from the restriction on partition J, this will prove the result.

Proposition 3.3.3 *Let* $m_I = \sum_J k(J)e_J$.

Then $\sum_I c_I[P^n]k(I) = (-1)^n(n+1)$.

<u>**Proof**</u>

Let $I \vdash n$ be the partition $1^{f_1}\cdots n^{f_n}$.

Then Girard's formula[1] gives us:

$$k(I) = (-1)^{n-F}\frac{n}{F}\frac{F!}{f_1!\cdots f_n!}$$

where $F = f_1 + \cdots + f_n$.

[1]See, for example, [9], volume 1, section 1.1.5. Although this formula seems to appear under the name of Girard in the literature, Muirhead in [11] indicates that it would be more correctly ascribed to Waring.

So,

$$\sum_I c_I[P^n]k(I) = \sum_I (-1)^{n-F}\frac{n}{F}\binom{-(n+1)}{1}^{f_1}\cdots\binom{-(n+1)}{n}^{f_n}\frac{F!}{f_1!\cdots f_n!} \qquad (3.7)$$

Now, if we indexed this sum over all ordered partitions, I_0, $\frac{F!}{f_1!\cdots f_n!}$ would be exactly the multiplicity of each term, so we have:

$$\begin{aligned}(3.7) &= \textstyle\sum_{I_0}(-1)^{n-F}\frac{n}{F}\binom{-(n+1)}{1}^{f_1}\cdots\binom{-(n+1)}{n}^{f_n} \\ &= \textstyle\sum_{j=1}^{n}(-1)^{n-j}\frac{n}{j}\sum_{l(I_0)=j}\binom{-(n+1)}{1}^{f_1}\cdots\binom{-(n+1)}{n}^{f_n}\end{aligned}$$

Lemma 3.3.4

$$\sum_{l(I_0)=j}\binom{-(n+1)}{1}^{f_1}\cdots\binom{-(n+1)}{n}^{f_n} = \sum_{k=0}^{j-1}(-1)^k\binom{j}{k}\binom{-(n+1)(j-k)}{n}$$

Proof

$\sum_{l(I_0)=j}\binom{-(n+1)}{1}^{f_1}\cdots\binom{-(n+1)}{n}^{f_n}$ = co-eff of x^n in $\{(1+x)^{-(n+1)}-1\}^j$

But, $\{(1+x)^{-(n+1)}-1\}^j = \sum_{k=0}^{j}(-1)^k\binom{j}{k}(1+x)^{-(n+1)(j-k)}$

The co-eff. of x^n in this is $\sum_{k=0}^{j-1}(-1)^k\binom{j}{k}\binom{-(n+1)(j-k)}{n}$ $\|$

So we have:

$$
\begin{aligned}
(3.7) &= \textstyle\sum_{j=1}^{n}(-1)^{n-j}\frac{n}{j}\sum_{k=0}^{j-1}(-1)^k\binom{j}{k}\binom{-(n+1)(j-k)}{n} \\
&= \textstyle\sum_{j=1}^{n}\sum_{k=1}^{j}(-1)^{n-k}\frac{n}{j}\binom{j}{k}\binom{-(n+1)k}{n}, \text{re-ordering the `}k\text{' terms} \\
&= \textstyle\sum_{k=1}^{n}\sum_{j=k}^{n}(-1)^{n-k}\frac{n}{j}\binom{j}{k}\binom{-(n+1)k}{n} \\
&= \textstyle\sum_{k=1}^{n}(-1)^{n-k}\binom{-(n+1)k}{n}n\sum_{j=k}^{n}\frac{1}{j}\binom{j}{k} \\
&= \textstyle\sum_{k=1}^{n}(-1)^{n-k}\binom{-(n+1)k}{n}\frac{n}{k}\sum_{j=k}^{n}\binom{j-1}{k-1} \\
&= \textstyle\sum_{k=1}^{n}(-1)^{n-k}\binom{-(n+1)k}{n}\frac{n}{k}\binom{n}{k} \\
&= (n+1)\textstyle\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{-(n+1)k-1}{n-1}
\end{aligned}
$$

Lemma 3.3.5

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{-(n+1)k-1}{n-1} = (-1)^n$$

<u>**Proof**</u>

Let $(*) = (1-x)^{-1}\{1-(1-x)^{-(n+1)}\}^n$.

$$
\begin{aligned}
\text{Now, } (*) &= \textstyle\sum_{k=0}^{n}(1-x)^{-1}(-1)^k\binom{n}{k}(1-x)^{-k(n+1)} \\
&= \textstyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1-x)^{-k(n+1)-1}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \text{co-eff. of } x^{n-1} \text{ in } (*) &= \textstyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{-(n+1)k-1}{n-1} \\
&= \textstyle\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{-(n+1)k-1}{n-1} + (-1)^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\text{But also, } (*) &= (1-x)^{-1}\{1-[1+o(x)]\}^n \\
&= \{1+o(x)\}\{o(x)\}^n \\
&= o(x^n)
\end{aligned}
$$

$\Rightarrow$ co-eff. of x^{n-1} in $(*)$ is 0.$\|$

This proves the proposition||.

This proves the theorem||.

3.4 Individual Examples

In the previous section we showed that the only odd primes to divide $\mathrm{hcf}\{c_I[M^n]|M^n \in MU(2n)\}$ were less than $l(I)+2$. In this section we examine some particular cases of simple chern numbers and show by example that in fact, with the exception of the case mentioned as equation (3.1), none of the odd primes less than $l(I)+2$ are involved in the h.c.f. for these numbers either. This provides perhaps the best method of computing the odd part of the h.c.f. for any particular case. The following calculations are given up to sign.

Example 3.4.1 ($c_1^2c_{n-2}$) *Theorem 3.3.2 tells us that the only odd prime that may be involved in the $hcf\{c_1^2c_{n-2}[M^n]|M^n \in MU(2n)\}$ is 3.*

To evaluate specific chern numbers we will use Lemma 1.4.5

1. $c_1^2c_{n-2}[(P^1)^n] = n(n-1)2^n$

This is not divisible by 3 if $n \equiv 2 \bmod 3$.

2.
$$\begin{aligned} c_1^2c_{n-2}[P^3\times(P^1)^{n-3}] &= (n-3)(n-4)c_{(0,0,3)}[P^3]c_1[P^1]^{n-3} \\ &\quad +(n-3)c_{(0,1,2)}[P^3]c_1[P^1]^{n-3} \\ &\quad +(n-3)c_{(1,0,2)}[P^3]c_1[P^1]^{n-3} \\ &\quad +c_{(1,1,1)}[P^3]c_1[P^3]^{n-3} \\ &= 2^{n-3}\{20(n-3)(n-4)+2.40(n-3)+64\} \\ &= 2^{n-1}(5n^2-15n+16) \end{aligned}$$

This is not divisible by 3 if $n \equiv 0 \bmod 3$.

3. Similarly, $c_1^2c_{n-2}[(P^3)^2 \times (P^1)^{n-6}] = 2^{n-2}.5(5n^2 - 25n + 42)$.

This is not divisible by 3 if $n \equiv 1 \bmod 3$.

These three cases cover every case except $c_1^2c_2[M^4]$, *but this is our one exception. So we can conclude, except for* $n = 4$, *no odd prime divides* $hcf\{c_1^2c_{n-2}[M^n] | M^n \in MU(2n)\}$.

Example 3.4.2 ($c_1^3c_{n-3}$) *Theorem 3.3.2 tells us that we require to eliminate the primes 3 and 5 from the hcf, the rest have been accounted for. As in the previous example we can compute specific chern numbers.*

1. $c_1^3c_{n-3}[P^2 \times (P^1)^{n-2}] = 2^{n-2}.3(n-2)(2n^2 - 5n + 6)$.

   This is not divisible by 5 for $n \equiv 0, 1, 3, 4 \bmod 5$.

2. $c_1^3c_{n-3}[P^3 \times (P^1)^{n-3}] = 2^{n-1}(5n^3 - 30n^2 + 73n - 68)$.

   This is not divisible by 5 for $n \equiv 0, 2, 3, 4 \bmod 5$.

   This is never divisible by 3.

These two cases together show us that no odd prime is involved in the $hcf\{c_1^3c_{n-3}[M^n] | M^n \in MU(2n)\}$.

Example 3.4.3 ($c_1c_2c_{n-3}$) *Similarly to the above, we require to eliminate the prime 3 from the hcf for this chern number.*

1. $c_1c_2c_{n-3}[P^3 \times (P^1)^{n-3}] = 2^{n-2}(5n^3 - 30n^2 + 77n - 76)$.

This is not divisible by 3 if $n \equiv 0, 2 \bmod 3$.

2. $c_1c_2c_{n-3}[P^4 \times (P^1)^{n-4}] = 2^{n-5}.5(7n^3 - 187n + 450)$.

   This is not divisible by 3 if $n \equiv 1 \bmod 3$

The above two cases show that, again except for the case $n = 4$, *no odd prime divides* $hcf\{c_1c_2c_{n-3}[M^n] | M^n \in MU(2n)\}$.

Although we only cover these three cases here, it can be readily seen that, with the help of theorem 3.3.2 any chern number can be considered in this manner.

Chapter 4

4.1 The Lattice of Partitions of Sets

In this section we shall present a view of symmetric functions first given by Peter Doubilet in [3] in 1972. In this paper Doubilet gives explicit interrelationships between symmetric functions using mobiüs inversion, and we repeat some of these formulae here. As an example of the power of this approach we shall also give a new direct proof of the result known as Girard's formula.

We will use the same notation for partitions of the number n that we have used up to now. In addition we will define:

$I! := i_1! \cdots i_r!$ if I is the partition $(i_1, \ldots, i_r)$

$|J| := f_1! \cdots f_n!$ if J is the partition $(1^{f_1} \ldots n^{f_n})$

We will consider the set $\Pi(D)$ of partitions of a finite set D. A partition π of D is a family of disjoint subsets $B_1, \ldots, B_r$, called blocks, whose union is D. In common with our notation for partitions of a number we will refer to π as being the partition $(B_1, \ldots, B_r)$.

The set $\Pi(D)$ can be ordered by putting $\sigma \leq \pi$ if every block of σ is contained in a block of π. This is, in fact, a partial ordering which makes $\Pi(D)$ into a lattice, with minimal element $\mathbf{0}= (\{d_1\}, \ldots, \{d_n\})$ and maximal element $\mathbf{1}= (\{d_1, \ldots, d_n\})$ if $D = \{d_1 \ldots, d_n\}$.

Given such a partition π of the set with n elements we write $\pi \vdash n$ and we assign to it the partition $\lambda(\pi)$ of the number n where the elements of $\lambda(\pi)$ are the number of elements in each block of π.

With this notation we can now quote the Mobiüs Inversion Theorem. This theorem is given in, for example, [1], Theorem 13.5, where the mobiüs function is also described. It should be noted that the partial order in [1] runs backwards from that used here.

Theorem 4.1.1 *Let $f(\pi)$ and $g(\pi)$ be real-valued functions with domain $\Pi(D)$, then:*

$$g(\pi_0) = \sum_{\pi \geq \pi_0} f(\pi)\ \forall \pi_0 \in \Pi(D) \Leftrightarrow$$

$$f(\pi_o) = \sum_{\pi \geq \pi_0} g(\pi)\mu(\pi, \pi_0)\ \forall \pi_0 \in \Pi(D)$$

And,

$$g(\pi_0) = \sum_{\pi \leq \pi_0} f(\pi)\ \forall \pi_0 \in \Pi(D) \Leftrightarrow$$

$$f(\pi_0) = \sum_{\pi \leq \pi_0} \mu(\pi_0, \pi) g(\pi)\ \forall \pi_0 \in \Pi(D)$$

Now, given a set D with n elements, consider $X = \{x_1, x_2, \ldots\}$ and let $F = \{f : D \to X\}$. For $f \in F$ its generating function $\gamma(f)$ is $\prod_{d \in D} f(d) = \prod_i x_i^{|f^{-1}(x_i)|}$. For $T \subseteq F$ the generating function $\gamma(T)$ is $\sum_{f \in T} \gamma(f)$. To any $f \in F$ we assign a partition $\ker f$ of D by putting d_1 and d_2 in the same block of $\ker f$ if and only if $f(d_1) = f(d_2)$.

We now consider two subsets of F:

$$\mathcal{M}_\pi = \{f \in F \mid \ker f = \pi\}$$

$$\mathcal{E}_\pi = \{f \in F \mid \ker f \wedge \pi = 0\}$$

Let $m_\pi = \gamma(\mathcal{M}_\pi)$ and $e_\pi = \gamma(\mathcal{E}_\pi)$, then [3] gives us the following result:

Theorem 4.1.2

$$m_\pi = |\lambda(\pi)| m_{\lambda(\pi)}$$

$$e_\pi = \lambda(\pi)! e_{\lambda(\pi)}$$

where $m_{\lambda(\pi)}$ *and* $e_{\lambda(\pi)}$ *are the monomial and elementary symmetric functions respectively.*

Using these functions Doubilet gives us the following result:

$$e_\pi = \sum_{\sigma \mid \sigma \wedge \pi = 0} m_\sigma \tag{4.1}$$

With a double application of mobiüs inversion this in turn yields the following:

$$m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{\mu(\mathbf{0}, \sigma)} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) e_\tau \tag{4.2}$$

We shall use the above formulae to examine chern numbers in the following section, but as an example of their original power we shall finish this section by providing a direct proof of Girard's formula, first referred to in the proof of Proposition 3.3.3 in Section 3.3.

Result 4.1.3 *The coefficient of* $e_{(n)}$ *in* m_I *is* $(-1)^{n-F} \frac{n}{F} \frac{F!}{|I|}$*, where* I *is the partition* $(1^{f_1} \ldots n^{f_n})$ *and* $f_1 + \cdots + f_n = F$.

Proof

Let $\pi \in \Pi(D)$ such that $\lambda(\pi) = I$. Then (4.2) gives us:

$$m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{\mu(\mathbf{0}, \sigma)} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) e_\tau$$

We are interested in the case $\tau = \mathbf{1} \Rightarrow \sigma = \mathbf{1}$, and we have:

coeff. of $e_\mathbf{1}$ in $m_\pi = \frac{\mu(\pi, \mathbf{1})}{\mu(\mathbf{0}, \mathbf{1})} \mu(\mathbf{1}, \mathbf{1})$

Now, from [1], Section 13.3, we can conclude:

$$\begin{aligned}
\mu(\pi,\mathbf{1}) &= (-1)^{-(F-1)}(F-1)! \\
\mu(\mathbf{0},\mathbf{1}) &= (-1)^{n-1}(n-1)! \\
\mu(\mathbf{1},\mathbf{1}) &= 1
\end{aligned}$$

So we have: coeff. of $e_{\mathbf{1}}$ in m_π is $(-1)^{n+F}\frac{(F-1)!}{(n-1)!}$.

$$\begin{aligned}
\text{But,}\quad m_\pi &= |\lambda(\pi)|m_{\lambda(\pi)} &&= |I|m_I \\
e_{\mathbf{1}} &= \lambda(\mathbf{1})!e_{\lambda(\mathbf{1})} &&= n!e_{(n)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \text{coeff. of } e_{(n)} \text{ in } m_I &= \tfrac{1}{|I|}(-1)^{n+F}\tfrac{(F-1)!}{(n-!)!}n! \\
&= (-1)^{n-F}\tfrac{n}{F}\tfrac{F!}{|I|}
\end{aligned}$$

4.2 An Abstract View of Chern Numbers of Projective Spaces

In this section we will apply the results of Section 4.1 to the study of chern numbers of projective spaces.

The basic motivation for what follows is a close examination of the polynomial:

$$c(P^n) = (1+x)^{-(n+1)}$$

The chern number $c_I[P^n]$ is the product of the coefficients of $x^{i_1}, \ldots, x^{i_r}$ in the above polynomial. It follows then that in the product:

$$(1+x_1)^{-(n+1)} \cdots (1+x_n)^{-(n+1)}$$

any chern number will be included as a suitable coefficient. To make this statement more precise we will need to use symmetric functions again.

Proposition 4.2.1

$$\sum_{I \vdash n} c_I[P^n] m_I = \sum_{K \vdash n} (-1)^{l(K)} \frac{\{n + l(K)\}!}{|K|} e_K$$

Proof

We shall look at the n-level set of:

$$(1+x_1)^{-(n+1)}\cdots(1+x_n)^{-(n+1)} \tag{4.3}$$

Now, the n-level set of (4.3) is a symmetric function and so, because the monomial symmetric functions form a basis of all symmetric functions, can be expressed as a sum of monomial symmetric functions. In this case:

coeff. of m_I = coeff. of $x_1^{i_1}\cdots x_r^{i_r} = \binom{-(n+1)}{i_1}\cdots\binom{-(n+1)}{i_r} = c_I[P^n]$, where I is the partition $(i_1,\ldots,i_r)$.

Hence we can say that the n-level set of (4.3) $= \sum_{I\vdash n} c_I[P^n]m_I(x_1,\ldots,x_n)$.

On the other hand we have:

$$\begin{aligned}(4.3) &= \{(1+x_1)\cdots(1+x_n)\}^{-(n+1)}\\ &= \{1+e_1(x_1,\ldots,x_n)+\cdots+e_n(x_1,\ldots,x_n)\}^{-(n+1)}\end{aligned}$$

$\Rightarrow$ n-level set $= \sum_{K\vdash n}\binom{-(n+1)}{f_1,\ldots,f_n}e_1^{f_1}\cdots e_n^{f_n}$,

where K is the partition $(1^{f_1}\ldots n^{f_n})$, $f_1+\cdots+f_n=F$ and:

$$\begin{aligned}\binom{-(n+1)}{f_1,\ldots,f_n} : &= \binom{-(n+1)}{f_1}\binom{-(n+f_1+1)}{f_2}\cdots\binom{-(n+F-f_n+1)}{f_n}\\ &= (-1)^{l(K)}\frac{\{n+l(K)\}!}{|K|}\end{aligned}$$

Hence, the n-level set of (4.3) $= \sum_{K\vdash n}(-1)^{l(K)}\frac{\{n+l(K)\}!}{|K|}e_K$.

This proves the result||.

We can therefore describe $c_I[P^n]$ by the following:

Corollary 4.2.2

$$\begin{aligned} c_I[P^n] &= \textit{coeff. of } m_I \textit{ in } \textstyle\sum_{K\vdash n}(-1)^{l(K)}\frac{\{n+l(K)\}!}{|K|}e_K \\ &= \textstyle\sum_{K\vdash n}(-1)^{l(K)}\frac{\{n+l(K)\}!}{|K|}(\textit{coeff of } m_I \textit{ in } e_K). \end{aligned}$$

We are now going to rewrite the above result in the language from [3], described in Section 4.1, to describe the lattice of partitions of sets.

First of all [3] gives us that:

$e_\kappa = \sum_{\pi|\pi\wedge\kappa=0} m_\pi$

$\Rightarrow \lambda(\kappa)!e_{\lambda(\kappa)} = \sum_{\pi|\pi\wedge\kappa=0} |\lambda(\pi)|m_{\lambda(\pi)}$

$\Rightarrow e_K = \sum_{I\vdash n} \frac{|I|}{K!}\#\{\pi|\lambda(\pi) = I, \pi\wedge\kappa = 0\}m_I$, where $\lambda(\kappa) = K$.

Secondly, in Corollary 4.2.2 we have a sum over all partitions $K \vdash n$ of a function of K. We will replace this with a sum over all partitions of a set, $\kappa \vdash n$, of a function of $\lambda(\kappa)$, but in this second summation each $\lambda(\kappa)$ will occur with a certain multiplicity. In fact we can see that each $\lambda(\kappa)$ will occur $\frac{n!}{\lambda(\kappa)!|\lambda(\kappa)|}$ times, and we must remember to take account of this as we change notation.

We can now write Corollary 4.2.2 as follows:

Proposition 4.2.3

$$c_I[P^n] = \frac{1}{I!}\sum_{\kappa|\kappa\wedge\pi=o}(-1)^{l(\kappa)}\frac{\{n+l(\kappa)\}!}{n!}$$

where $\lambda(\pi) = I$.

Proof

Fix π such that $\lambda(\pi) = I$

Then:

$$\begin{aligned}
c_I[P^n] &= \textstyle\sum_{K\vdash n}(-1)^{l(K)}\frac{\{n+l(K)\}!}{|K|}(\text{coeff of } m_I \text{ in } e_K) \\
&= \textstyle\sum_{\kappa\vdash n}\frac{\lambda(\kappa)!|\lambda(\kappa)|}{n!}(-1)^{l(\kappa)}\frac{\{n+l(\kappa)\}!}{|\lambda(\kappa)|}\frac{|I|}{\lambda(\kappa)!}\#\{\pi|\lambda(\pi) = I, \ \pi\wedge\kappa = 0\} \\
&= \textstyle\sum_{\pi|\lambda(\pi)=I}\sum_{\kappa|\kappa\wedge\pi=0}\frac{|I|}{n!}(-1)^{l(\kappa)}\frac{\{n+l(\kappa)\}!}{n!} \\
&= \textstyle\frac{n!}{I!|I|}\sum_{\kappa|\kappa\wedge\pi=0}\frac{|I|}{n!}(-1)^{l(\kappa)}\frac{\{n+l(\kappa)\}!}{n!} \\
&= \textstyle\frac{1}{I!}\sum_{\kappa|\kappa\wedge\pi=0}(-1)^{l(\kappa)}\frac{\{n+l(\kappa)\}!}{n!} \,\|
\end{aligned}$$

This gives us a second presentation of the chern number $c_I[P^n]$, the first being given in Lemma 1.4.7. As we shall show this new presentation lends itself naturally to looking at not just single projective spaces but at arbitrary cross products of projective spaces.

However, before this we shall demonstrate the practicality of this language by calculating an example.

Example 4.2.4 *What is* $c_{(3,1)}[P^4]$ *?*

Fix $\pi = (123)(4)$, *then* $I! = 3!1! = 6$.

If $\kappa = (14)(3)(2)$ *then* $(-1)^{l(\kappa)}\frac{\{4+l(\kappa)\}!}{4!} = -\frac{7!}{4!} = -5.6.7$

Similarly for $\kappa = (1)(24)(3)$ *or* $(1)(2)(34)$.

If $\kappa = (1)(2)(3)(4)$ *we have* $+\frac{8!}{4!} = 5.6.7.8$

So $c_{(3,1)}[P^4] = \frac{1}{6}(5.6.7.8 - 3.5.6.7) = 5.5.7$

This concurs with the value for $c_{(3,1)}[P^4]$ *computed directly.*$\|$

Having obtained this result for a single projective space we shall now continue to examine an arbitrary cross product of projective spaces: P^J, where J is the partition $(j_1, \ldots, j_s)$.

From 1.4.5 we have:

$$\begin{aligned} c_I[P^J] &= \sum_{I_1+\cdots+I_s=I, I_k\vdash j_k} c_{I_1}[P^{j_1}]\cdots c_{I_s}[P^{j_s}] \\ &= \sum \prod_{k=1}^{s} c_{I_k}[P^{j_k}] \\ &= \sum \prod_{k=1}^{s} \left\{ \frac{1}{I_k!} \sum_{\kappa\wedge\pi_k=0} (-1)^{l(\kappa)} \frac{(j_k+l(\kappa))!}{j_k!} \right\} \end{aligned}$$

Where here, $\lambda(\pi_k) = I_k$.

Now, we want to change the variables of summation, namely the $I_1, \ldots, I_s$, to partitions of sets. We can do this in the following way.

Fix $\sigma = (S_1, \ldots, S_r)$ such that $\lambda(\sigma) = I$.

For any $\tau = (T_1, \ldots, T_s)$ where $\lambda(\tau) = J$ we have:

$I_k := \lambda(T_k \cap S_1, \ldots, T_k \cap S_r)$.

Now, if we sum over all $\tau|\lambda(\tau) = J$ we will consider each $I_1 + \ldots + I_s$ a total of $\frac{I!}{I_1!\cdots I_s!}$ times.

But, because of possible duplicity in the numbers $j_1, \ldots, j_s$, in the summation $\sum_{I_1+\cdots+I_s=I}$ we consider each particular element $|J|$ times. So we have:

$$\begin{aligned} c_I[P^J] &= \sum_{\lambda(\tau)=J} \frac{|J|I_1!\cdots I_s!}{I!} \prod_{k=1}^{s} \frac{1}{I_k!} \sum_{\kappa\wedge\pi_k=0} (-1)^{l(\kappa)} \frac{(j_k+l(\kappa))!}{j_k!} \\ &= \sum_{\lambda(\tau)=J} \frac{|J|}{I!} \prod_{k=1}^{s} \sum_{\kappa\wedge\pi_k=0} (-1)^{l(\kappa)} \frac{(j_k!+l(\kappa))!}{j_k!} \end{aligned}$$

We shall now note the following general result:

$$\prod_{k=1}^{s}\sum_{a\in A} f(k,a) = \sum_{a_1,\ldots,a_s\in A^s}\prod_{k=1}^{s} f(k,a_k)$$

So we have:

$$\begin{aligned} c_I[P^J] &= \textstyle\sum_{\lambda(\tau)=J}\frac{|J|}{I!}\sum_{\kappa_1,\ldots,\kappa_s;\kappa_k\wedge\pi_k=0}\prod_{k=1}^{s}(-1)^{l(\kappa_k)}\frac{(j_k+l(\kappa_k))!}{j_k!} \\ &= \textstyle\frac{|J|}{I!}\sum_{\lambda(\tau)=J}\sum_{\kappa\leq\tau;\kappa\wedge\sigma=0}\prod_{k=1}^{s}(-1)^{l(\kappa_k)}\frac{(j_k+l(\kappa_k))!}{j_k!} \end{aligned}$$

Where the condition that $\kappa \leq \tau; \kappa \wedge \sigma = 0$ can be seen to be equivalent to the conditions that κ is divided into $\kappa_1, \ldots, \kappa_s$ and each $\kappa_k \wedge \pi_k = 0$. We shall write this as the following result:

Theorem 4.2.5

$$c_I[P_J] = \frac{|J|}{I!}\sum_{\lambda(\tau)=J}\sum_{\kappa\leq\tau;\kappa\wedge\sigma=0}\prod_{k=1}^{s}(-1)^{l(\kappa_k)}\frac{(j_k+l(\kappa_k))!}{j_k!}$$

Where $\lambda(\sigma) = I$ and κ_k is the particular partition of the k'th block of τ, seen as a part of the partition κ.

This presentation gives a formula for the general chern number $c_I[P^J]$ purely as a function of the partitions I and J and the partitions of the set of n objects with I and J as their underlying partitions. It should be noted that by using the language of the lattice of partitions of sets we obtain what can be seen as a more natural summation than that in Lemma 1.4.5.

Finally we present two examples of how this new approach can be used: firstly to make calculations in a systematic way and secondly to provide some interesting information in special cases.

Example 4.2.6 *What is $c_{2,2}[P^3 \times P^1]$?*

Let $\sigma = (12)(34)$.

$\frac{|J|}{I!} = \frac{1!1!}{2!2!} = \frac{1}{4}$.

Consider $\tau = (123)(4)$. We want $\kappa \leq \tau$; $\kappa \wedge \sigma = 0$.

If $\kappa = (13)(2)(4)$ then $\prod_{k=1}^{s}(-1)^{l(\kappa)}\frac{(j_k+l(\kappa))!}{j_k!} = +\frac{5!}{3!} \times -\frac{2!}{1!} = -40$

Similarly if $\kappa = (1)(23)(4)$ we have -40.

If $\kappa = (1)(2)(3)(4)$ we have $-\frac{6!}{3!} \times \frac{2!}{1!} = +240$.

We get the same numbers if $\tau = (124)(3)$ or $(134)(2)$ or $(234)(1)$.

So in total we get $\frac{1}{4}.4.(240 - 40 - 40) = +160$.

This corresponds with the correct value for $c_{2,2}[P^3 \times P^1]$ computed directly.||

Example 4.2.7 *We shall assume each of the dimensions of the projective spaces, namely the j_k's, are of the form $p^{t(k)} - 2$ where p is the particular prime we are interested in and $t(k) \geq 1$ for each k.*

Now, if this is the case then we can see that:

$$(-1)^{l(\kappa_k)}\frac{(j_k + l(\kappa_k))!}{j_k!} \equiv \left\{ \begin{array}{ll} -(j_k+1) \equiv 1 & \text{if } l(\kappa_k) = 1 \\ 0 & \text{otherwise} \end{array} \right\} \pmod{p}$$

So we have:

$$
\begin{aligned}
c_I[P^J] &\equiv \tfrac{|J|}{I!}\textstyle\sum_{\lambda(\tau)=J}\#\{\tau|\tau\wedge\sigma=0\} \pmod p\\
&= \tfrac{|J|}{I!}\#\{\tau|\lambda(\tau)=J,\ \tau\wedge\sigma=0\}\\
&= \textit{coeff of } m_J \textit{ in } e_I\\
&= \textit{coeff of } m_I \textit{ in } e_J
\end{aligned}
$$

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